Complete Metric Space

The aim of this chapter is to study one of the properties of metric space. The notion of distance between points of an abstract set leads naturally to the discussion of convergence of sequences and Cauchy sequences in the set. Unlike the situation of real numbers, where each Cauchy sequence is convergent, there are metric spaces in which Cauchy sequences fail to converge. A metric space in which every Cauchy sequence converges is called a 'complete metric space'. This property plays a vital role in analysis when one wishes to make an existence statement. We shall see that a metric space need not be complete and hence we shall find conditions under which such a property can be ensured.

1 Sequence

Let (X, d) be a metric space. A *sequence of points in* X is a function $x : \mathbb{N} \to X$. In other words, a sequence assigns to each $n \in \mathbb{N}$ a uniquely determined element of X. If $f(n) = x_n$, it is customary to denote the sequence by the symbol $\langle x_n \rangle$, or by $x_1, x_2, \cdots, x_n, \cdots$.

(i) Let $\langle x_n \rangle$ be a sequence in the set X. If there exists $n_0 \in \mathbb{N}$ such that $x_n = a$ for all $n > n_0$, then the sequence $\langle x_n \rangle$ is called *eventually constant*.

(ii) If $x_n = n$ for all $n \in \mathbb{N}$, then the sequence $\langle x_n \rangle$ is called *constant sequence*. Obviously, a constant sequence is a special case $(n_0 = 1)$ of an eventually constant sequence.

(iii) A sequence $\langle x_n \rangle$ is said be *frequently in a set* S if for each positive integer $m \in \mathbb{N}$, there exists a positive integer $n \geq m$ such that $x_n \in S$.

(iv) Suppose X is a non-empty set and $x = \langle x_n \rangle$ is a sequence in X. For each $m \in \mathbb{N}$, the set $\{x_n : n \in \mathbb{N}, n \geq .m\}$ is called the m^{th} *tail of the sequence* $\langle x_n \rangle$.

1.1 Convergent Sequence

The concept of a convergent sequence plays an important role to investigate the closedness of a set, the continuity of a function and several other properties. In this section, we give an introduction to the convergence of sequences and the properties under which a sequence is convergent in arbitrary metric spaces.

Definition 1 [**Convergence of a Sequence**] Let (X, d) be a metric space and A be a non-empty subset of X. A sequence $\langle x_n \rangle$ in X is said to be *convergent* if there is a point $a \in X$ such that for each real $\varepsilon > 0$, $\exists$ a $k \in \mathbb{N}$ such that $d(x_n, a) < \varepsilon \ \forall \ n \geq k$. Equivalently for each open sphere $S(a, \varepsilon)$ centred at a, $\exists$ a positive integer k such that

$$x_n \in S(a, \varepsilon); \quad \forall n \geq k. \tag{1}$$

Here a is called the *limit of the sequence* $\langle x_n \rangle$ and we write $\lim\limits_{n \to \infty} x_n = a$.

(i) Therefore a sequence $\langle x_n \rangle$ in a matric space (X, d) converges to a if and only if the $\langle d(x_n, a)\rangle\rangle$ in $\mathbb{R}_u$ converges to zero.

(ii) A singleton set in a metric space is, of course, included in every ball centred at its only point. A sequence that has a singleton set for a tail is said to be eventually constant; such sequences must converge in any metric space to which they belong. Thus, an eventually constant sequence, and hence a constant sequence, is convergent.

(iii) A constant sequence, in which all terms are the same, is a special case of an eventually constant sequence and converges to its single value in any metric space to which it belongs.

(iv) In a discrete metric space X, it is very difficult for a sequence to converge. Let (X, d) be a discrete metric space, and $\langle x_n \rangle$ be a sequence in X, which converges to a in X. Let $\varepsilon = 1/2$. Since $\langle x_n \rangle$ converges to a, $\exists m \in \mathbb{N}$ such that $d(x_n,) < \varepsilon = 1/2$ for all $n \geq m$. This means that $x_n = a$ for $n \geq m$, i.e., the sequence is of the form $\langle x_1, x_2, \cdots, x_{m-1}, a, a, \cdots \rangle$. Each singleton set $\{a\}$ is open and the only way that $\{a\}$ can include a tail of a sequence $\langle x_n \rangle$ is if the sequence is eventually constant with $x_n = a$ for all sufficiently large $n \in N$. Thus in a discrete metric space, a sequence can converge to a point only if it is an eventually constant sequence.

(v) The sequence $\langle 1/n \rangle$ of inverses of the natural numbers converges to 0 because for each $r \in \mathbb{R}^+$ there exists $k \in \mathbb{N}$ such that $1/k < r$ and then the ball $B[0; r)$ includes the k^{th} tail of the sequence $\langle 1/n \rangle$.

(vi) Let X denote the space of all sequences of numbers with metric d defined by

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \; \frac{|x_j - y_j|}{1 + |x_j - y_j|}; \quad x = \langle x_j \rangle, y = \langle y_j \rangle \in X.$$

Let $\langle x^{(n)} \rangle = \langle \langle x_j^{(n)} \rangle \rangle$ be a sequence in X which converges to, say, $x \in X$. In other words,

$$d(x^{(n)}, x) = \sum_{j=1}^{\infty} \frac{1}{2^j} \; \frac{|x_j^{(n)} - x_j|}{1 + |x_j^{(n)} - x_j|} \to 0 \text{ as } n \to \infty.$$

This means that, for $\varepsilon > 0$, there exists an integer $n_0(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{j=1}^{\infty} \frac{1}{2^j} \; \frac{|x_j^{(n)} - x_j|}{1 + |x_j^{(n)} - x_j|} < \varepsilon, \text{ whenever } n \geq n_0(\varepsilon)$$

$$\Rightarrow \quad \frac{1}{2^j} \; \frac{|x_j^{(n)} - x_j|}{1 + |x_j^{(n)} - x_j|} < \varepsilon, \text{ whenever } n \geq n_0(\varepsilon); j = 1, 2, \cdots.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\lim\limits_{n\to\infty} x_j^{(n)} = x_j$ for each j. Consider any $\varepsilon > 0$. There exists an integer m such that $\sum\limits_{j=m+1}^{\infty} 1/2^j < \varepsilon/2$. Consequently,

$$\begin{aligned} d\Big(x^{(n)}, x\Big) &= \sum_{j=1}^{m} \frac{1}{2^j} \; \frac{|x_j^{(n)} - x_j|}{1 + |x_j^{(n)} - x_j|} + \sum_{j=m+1}^{\infty} \frac{1}{2^j} \; \frac{|x_j^{(n)} - x_j|}{1 + |x_j^{(n)} - x_j|} \\ &< \sum_{j=1}^{m} \frac{1}{2^j} \; \frac{|x_j^{(n)} - x_j|}{1 + |x_j^{(n)} - x_j|} + \frac{\varepsilon}{2}. \end{aligned}$$

As the first sum contains only finitely many terms and $\lim\limits_{n\to\infty} x_j^{(n)} = x_j$ for each j, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{j=1}^{m} \frac{1}{2^j} \frac{|x_j^{(n)} - x_j|}{1 + |x_j^{(n)} - x_j|} < \varepsilon/2, \text{ whenever } n \geq n_0(\varepsilon).$$

Hence, $d(x^{(n)}, x) < \varepsilon$ whenever $n \geq n_0(\varepsilon)$. Thus, convergence in the space of all sequences is co-ordinatewise convergence. The metric space is known as *Frechet space.*

RESULT 1 **(Criteria for Convergence):** Suppose X is a metric space, $a \in X$ and $\langle x_n \rangle$ is a sequence in X. Then following are some equivalent statements, any one of which might be used in a definition of convergence in metric spaces:

(i) *(closure criterion I)*: $\bigcap \Big\langle \overline{\Big\langle x_n : n \in S \Big\rangle} \Big|\ S \subseteq \mathbb{N}, S = \text{ infinite} \Big\rangle = \langle a \rangle.$

(ii) *(closure criterion II)*: $x \in \bigcap \Big\langle \overline{\Big\langle x_n : n \in S \Big\rangle} \Big|\ S \subseteq \mathbb{N}, S = \text{ infinite} \Big\rangle.$

(iii) *(distance criterion)* : $\text{dist} \Big(a, \{x_n : n \in S\} \Big) = 0$ for every infinite subset S of $\mathbb{N}$.

(iv) *(ball criterion)*: Every open ball centred at a includes a tail of $\langle x_n \rangle$.

(v) *(open set criterion)*: Every open subset of X that contains a includes a tail of $\langle x_n \rangle$.

THEOREM 1 **In a metric space, every convergent sequence has an unique limit.**

Proof: Let (X, d) be a metric space and $\langle x_n \rangle$ be a convergent sequence in X. If possible, let a and b be two limits of $\langle x_n \rangle$, i.e., $a \neq b$, so $d(a, b) > 0$. Since $\lim\limits_{n\to\infty} x_n = a$ and $\lim\limits_{n\to\infty} x_n = b$, so corresponding to an $\varepsilon > 0$, $\exists\ n_1, n_2 \in \mathbb{N}$ such that

$$d(x_n, a) < \varepsilon/2, \text{ for } n \geq n_1 \text{ and } d(x_n, b) < \varepsilon/2, \text{ for } n \geq n_2.$$

Using the triangle inequality, we have

$$\begin{aligned} & d(x_n, x_n) \leq d(x_n, a) + d(a, b) + d(x_n, b) \\ \text{or ,} \quad & d(x_n, x_n) - d(a, b) \leq d(x_n, b) + d(x_n, a). \end{aligned} \tag{2}$$

Also, interchanging x_n, a and x_n, b in Eq.(2), we obtain

$$d(a, b) - d(x_n, x_n) \leq d(x_n, b) + d(x_n, a). \tag{3}$$

Combining (2) and (3), we get

$$\begin{aligned} & -\Big[d(x_n, b) + d(a, x_n)\Big] \leq -\Big[d(x_n, x_n) - d(a, b)\Big] \leq d(x_n, b) + d(x_n, a) \\ \Rightarrow \quad & \Big|d(x_n, x_n) - d(a, b)\Big| \leq d(x_n, b) + d(x_n, a) \\ \Rightarrow \quad & 0 \leq \Big|d(x_n, x_n) - d(a, b)\Big| \leq \varepsilon, \text{ for } n \geq m = \max\{n_1, n_2\} \\ \Rightarrow \quad & 0 = d(x_n, x_n) \to d(a, b). \end{aligned}$$

$\therefore d(a, b) = 0$ and so $a = b$. Thus the limit of the sequence $\langle x_n \rangle$ is unique.

Note: A sequence $\langle x_n \rangle$ which is not convergent is called *divergent*.

1.2 Convergence of Subsequences

Every subsequence of a convergent sequence converges to the same limit as the parent sequence. A sequence that does not converge, however, may have many convergent subsequences with various limits. We discover in this section what those limits are.

Definition 2 Suppose X is a metric space, $x \in X$ and $\langle x_n \rangle$ is a sequence in X that converges to x. Suppose $\langle x_{m_n} \rangle$ is a subsequence of $\langle x_n \rangle$. Then the sequence $\langle x_{m_n} \rangle$ also converges to x.

THEOREM **2 Let $\langle x_n \rangle$ be a convergent sequence in a metric space (X, d) such that $x_n \to x$ as $n \to \infty$. If $\langle x_{n_k} \rangle$ is any subsequence of $\langle x_n \rangle$ then $x_{n_k} \to a$ as $k \to \infty$.**

Proof: Since $\langle x_n \rangle$ converges to a for any $\varepsilon > 0, \exists$ a natural number m such that

$$d(x_n, a) < \varepsilon; \qquad \forall\, n \geq m.$$

Since $\langle n_k \rangle$ is a strictly monotonically sequence of natural number so $\exists$ a $k_1 \in \mathbb{N}$ such that $n_k \geq m, \forall\, k \geq k_1$. Form these, we get

$$d(x_{n_k}, a) \leq d(x_{n_k}, x_n) + d(x_n, a) < \varepsilon, \qquad \forall\, k \geq k_1.$$

$\therefore$ The sub-sequence $\langle x_{n_k} \rangle$ converges to a. Thus every subsequence of a convergent sequence is convergent.

RESULT **2** If a subsequence in a metric space (X, d) is convergent, then the sequence itself need not be convergent. for example

Consider $\langle x_n = (-1)^n; n \in \mathbb{N} \rangle$ in $\mathbb{R}_u$. Let the subsequence $\langle x_{2n} \rangle$ of the sequence $\langle x_n \rangle$ as $x_{2n} = 1; \forall n \in \mathbb{N}$, then $x_{2n} \to 1$ as $n \Rightarrow \infty$ in $\mathbb{R}_u$. However $\langle x_n \rangle$ is not convergent sequence.

1.3 Bounded Sequence

Definition 3 In a matric space a sequence is said to be *bounded* if the range of the sequence forms a bounded set.

THEOREM **3 In a matric space every convergent sequence is bounded.**

Proof: Let $\langle x_n \rangle$ be a convergent sequence to a. Then for$\varepsilon = 1, \exists$ a natural number k such that

$$d(x_n, a) < 1, \forall\, n \geq k.$$

Let $r = \max\{1, d(x_n, x) : 1 \leq n < k\}$, then $d(x_n, x) \leq r, \forall\, n \in \mathbb{N}$. Now

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x) + d(x, x_n) \\ &\leq r + r = 2r. \end{aligned}$$

i.e., $d(x_m, x_n) \leq 2r, \forall\, m, n \in \mathbb{N}$. The diameter of the range set of the sequence is bounded by $2r$. Thus $\langle x_n \rangle$ is bounded.

RESULT **3** The property of convergence of a sequence in a metric space (X, d) depends on the set X as well as the metric d.

(i) When different metrics are placed on a given set, the sequences that converge may differ and limits may differ too. For example, when $\mathbb{R}$ is endowed with the discrete metric, the sequence $\langle 1/n \rangle$ does not converge. However, the formal definition of convergence (unlike the distance criterion and the ball criterion) does not rely on a fixed metric but only on the open sets produced by the metric. It follows that, on any given set, metrics that produce the same topology produce also the same convergent sequences with the same limits.

(ii) **Convergence depends on set** X: Let $\langle x_n \rangle$ be a sequence in the usual metric space $\mathbb{R}_u$, where $x_n = \frac{1}{n}$, $\forall\, n \in \mathbb{N}$. Then $\langle x_n \rangle$ converges to 0 in $\mathbb{R}_u$. However if we take $X = (0, 1]$ with usual matric, then the same sequence does not converge in X, as '0' does not belongs to X.

(iii) **Convergence depends on the metric** d: We know the set $C[0, 1]$ of all continuous function over $[0, 1]$ forms a matric space with respect to d_1 as well as d_∞ where

$$d_1(x, y) = \int_0^1 |x(t) - y(t)| dt$$
$$\text{and} \quad d_\infty(x, y) = \sup_{t \in [0,1]} |x(t) - y(t)| \qquad x, y \in [0, 1].$$

Let us consider the sequence $\langle x_n \rangle$ in the space $C[0, 1]$, where $x_n(t) = e^{-nt}$, $t \in [0, 1]$, $n \in \mathbb{N}$. Then the sequence $\langle x_n \rangle$ converges to zero in $C[0, 1]$ with respect to the metric d_1, since

$$\begin{aligned} d_1(x_n, 0) &= \int_0^1 |x_n(t) - 0| dt = \int_0^1 |e^{-nt} - 0| dt \\ &= \left[-\frac{1}{n} e^{-nt}\right]_0^1 = \frac{1}{n}(1 - e^{-n}) \to 0. \end{aligned}$$

On the other hand the same sequence does not converge to zero, with respect to the metric d_∞, since

$$\begin{aligned} d_\infty(x, y) &= \sup_{t \in [0,1]} |x_n(t) - 0| = \sup_{t \in [0,1]} |e^{-nt} - 0| \\ &= \sup_{t \in [0,1]} e^{-nt} = 1 \nrightarrow 0 \qquad \text{as } n \to \infty \end{aligned}$$

Theorem 4 **(Characterization of limit points of a set in terms of convergent sequence) : Let (X, d) be a metric space, $A \subseteq X$ and $a \in X$.**

(i) Then a is a limit point of A if and only if $\exists$ a sequence $\langle x_n \rangle$ of points of A, none of which equals to a such that $\lim_{n \to \infty} x_n = a$.

(ii) The set A is closed if and only if every convergent sequence of points of A has its limit in A.

Proof: Let (X, d) be a metric space, $A \subseteq X$.

(i) Suppose $a \in X$ is a limit point of A. Then $\left[S(x, r) - \{a\}\right] \cap A \neq \phi$ for each $r > 0$. Let us construct a sequence $\langle x_n \rangle$ where,

$$x_n \in \left[\left\{S\left(a, \frac{1}{n}\right) - \{a\}\right\} \cap A\right]; \text{ for each } n \in \mathbb{N}.$$

Then $x_n \neq a; \quad \forall n \in \mathbb{N}$. We now prove that $\langle x_n \rangle$ converges to a. Let $\varepsilon > 0$ be given by Archimidean property $\exists$ a natural number $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$. Then for $n > k$,

$$\begin{aligned} & d(x_n, a) < \frac{1}{n} < \frac{1}{k} < \varepsilon; \qquad \text{as } x_n \in S(a, \frac{1}{n}) \\ \text{i.e.,} \quad & d(x_n, a) < \varepsilon \qquad \forall\, n \in \mathbb{N}. \end{aligned}$$

Thus $\{x_n\}$ converges to a.

Conversely, suppose $\exists$ a sequence $\langle x_n \rangle$ in A none of which equals to a such that $x_n \to a$ as $n \to \infty$. We shall prove that a is a limit point of A. Let $r > 0$ be arbitrary. Since $\langle x_n \rangle$ converges to a, $\exists$ a natural number k such that $d(x_n, a) < r, \forall\, n \geq k$ i.e.,

$$d(x_k, a) < r \Rightarrow x_k \in S(a, r).$$

Since $x_k \neq a$ and $x_k \in A$,, so

$$\begin{aligned} & x_k \in \big[\big(S(a, r) - \{a\}\big) \cap A\big] \\ \Rightarrow \quad & \big(S(a, r) - \{a\}\big) \cap A \neq \phi. \end{aligned}$$

Since $r > 0$ is arbitrary, a is a limit point of A.

(ii) Suppose that A is closed and $\langle x_n \rangle$ is a sequence of points of A which converges to a point $a \in X$. We have to show that $a \in A$.

(a) If the range set of the sequence $\langle x_n \rangle$ is infinite, then it follows from part (i) that a is a limit point of this set. Since A is closed, we have $a \in A$.

(b) If the range set of the sequence $\langle x_n \rangle$ is finite, then $x_n = a$ for all $n \geq n_0$, since $\langle x_n \rangle$ is a convergent sequence. Since each term of the sequence $\langle x_n \rangle$ belongs to A, so is a.

Conversely, assume that every convergent sequence of points of A converges to a point of A. We are to show that A is closed by showing that it contains all its limit points.

Let a be a limit point of A. Then by part (i), there is a sequence $\langle x_n \rangle$ of points of A, none of which equals to a, such that $x_n \to a$. By hypothesis $a \in A$, so A is closed.

THEOREM 5 **Let $\langle x_n \rangle$ be a sequence in a matric space (X, d) which converges to x in X and A be the range of $\langle x_n \rangle$**

(i) If A is a finite set then $x_n = x$ for infinitely many n.

(ii) If A is an infinite set then x is a limit point of A.

Proof:

(i) Suppose range of A is finite and $A = \{y_1, y_2, \cdots, y_m\}$. We first show that $a \in A$. If possible let $a \notin A$, then $x \neq y_i$ for $i = 1, 2, 3, \cdots, m$. Then $d(y_i, a) > 0$ for $i = 1, 2, 3, \cdots, m$. Let $r = \min\{d(y_i, a) : 1 \leq i \leq m\}$. Then $y_i \notin S(a, r)$ for all $1 \leq i \leq m \Rightarrow d(x_n, a) \geq r$ for all n, which is a contradiction, as sequence $\langle x_n \rangle$ converges to a.

$\therefore\ a \in A$. With out loss of generality suppose $a = y_1$. Then $d(y_1, y_i) > 0$ for $2 \leq i \leq m$.

Suppose $r_1 = \min\{d(y_1, y_i);\ 2 \le i \le m\}$. So, $y_i \notin S(y_1, r_1)$ for $i = 1, 2, 3, \cdots, m; d(y_1, y_i) > r_1$. Then $r_1 > 0$. Again since $\langle x_n \rangle$ converges to $x = y_1$, for $r_1 > 0$, $\exists$ a natural number k such that

$$\begin{aligned} & d(x_n, a) < r_1;\quad \forall n \ge k \\ \Rightarrow\quad & d(x_n, a) = 0;\quad \forall n \ge k \qquad [\text{Since } d(x_n, a) = d(y_i, y_1) \text{ either } = 0 \text{ or } \ge r_1] \\ \Rightarrow\quad & x_n = a;\quad \forall n \ge k \\ \text{i.e.}\quad & x_n = a \qquad \text{for infinitely many values of } n. \end{aligned}$$

(ii) Suppose A is an infinite set. Let $r > 0$. Since $\langle x_n \rangle$ converges to a for $r > 0$, $\exists$ a natural number k such that

$$d(x_n, a) > r \Rightarrow x_n \in S(a, r), \quad \forall n \ge k.$$

Since A is an infinite set, $\exists$ a natural number $m \ge k$ such that $x_m \neq a$. So

$$\begin{aligned} & x_m \in \big(S(a, r) - \{a\}\big) \\ \Rightarrow\quad & x_m \in \big(S(a, r) - \{a\}\big) \cap A \\ \Rightarrow\quad & \big(S(a, r) - \{a\}\big) \cap A \neq \phi \end{aligned}$$

Since $r > 0$ is arbitrary, so x is a limit point of A.

THEOREM 6 **If $\langle x_n \rangle$ and $\langle y_n \rangle$ are two sequences in a metric space (X, d) such that $\langle x_n \rangle$ converges to a and $\langle y_n \rangle$ converges to b then $d(x_n, y_n) \to d(a, b)$ as $n \to \infty$.**

Proof: Since $\langle x_n \rangle$ and $\langle y_n \rangle$ converge to a and b respectively, for any $\varepsilon > 0$, $\exists$ natural numbers k_1 and k_2 such that

$$d(x_n, a) < \varepsilon/2;\ \forall\ n \ge k_1;\ \text{and } d(y_n, b) < \varepsilon/2;\ \forall\ n \ge k_2.$$

Let $k = \max\{k_1, k_2\} \in \mathbb{N}$, then

$$\begin{aligned} \Big| d(x_n, y_n) - d(a, b) \Big| &\le d(x_n, a) + d(y_n, b) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon;\ \forall n \ge k. \end{aligned}$$

Thus $d(x_n, y_n) \to d(a, b)$ as $n \to \infty$.

1.4 Convergence in Product Spaces

Let us turn now to convergence in product spaces.

THEOREM 7 **Let (X, d_1) and (Y, d_2) be two metric spaces. A sequence $\langle (x_n, y_n) \rangle$ in the product metric space $X \times Y$ converges to (a, b) if and only if the sequence $\langle x_n \rangle$ converges to a in X and $\langle y_n \rangle$ converges to b in Y.**

Proof: Consider the product metric $d : X \times Y \to \mathbb{R}$ given by

$$d\Big((x_1, y_1), (x_2, y_2)\Big) = d_1(x_1, x_2) + d_2(y_1, y_2);\ \ (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Clearly, $(X \times Y, d)$ is a metric space. Assume that first $x_n \to a$ in X and $y_n \to b$ in Y. Then, for each $\varepsilon > 0$, $\exists$ positive integers $n_1, n_2 \in \mathbb{N}$ such that

$$d_1(x_n, a) < \varepsilon/2, \forall n \geq n_1 \text{ and } d_1(y_n, b) < \varepsilon/2, \forall n \geq n_2.$$

Let $n_0 = \max\{n_1, n_2\} \in \mathbb{N}$. Then

$$\begin{aligned} d\Big((x_n, y_n), (a, b)\Big) &= d_1(x_n, a) + d_2(y_n, b) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad \forall n \geq n_0. \end{aligned}$$

Thus $(x_n, y_n) \to (a, b)$. Conversely, let $(x_n, y_n) \to (a, b)$ in $X \times Y$. Then, for each $\varepsilon > 0$, $\exists$ a positive integer $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} & d\Big((x_n, y_n), (a, b)\Big) < \varepsilon, \quad \forall n \geq n_0 \\ \Rightarrow \quad & d_1(x_n, a) + d_2(y_n, b) < \varepsilon, \quad \forall n \geq n_0 \\ \Rightarrow \quad & d_1(x_n, a) < \varepsilon, d_2(y_n, b) < \varepsilon, \quad \forall n \geq n_0 \\ \Rightarrow \quad & x_n \to a \text{ and } y_n \to b. \end{aligned}$$

1.5 Cauchy Sequence

In real analysis, we must have come across the concept of a Cauchy sequence where it is proved that a sequence is a Cauchy sequence if and only if it is convergent. Such an equivalence is no longer true when these notions are extended to metric spaces. Nevertheless, the notion of Cauchy sequence is important in metric spaces. For an arbitrary topological space, the concept of Cauchy sequence does not exist.

Definition 4 Let (X, d) be a metric space. A sequence $\langle x_n \rangle$ of elements of X is said to be a *fundamental* or *Cauchy sequence* if for any $\varepsilon > 0$, $\exists\ k \in \mathbb{N}$ such that

$$d(x_m, x_n) < \varepsilon \qquad \forall\ m, n \geq k. \tag{4}$$

Equivalently, let $\langle x_n \rangle$ be a sequence and let $T_n = \{x_k : k \geq n\}, n \in \mathbb{N}$. Then $\langle x_n \rangle$ is a *Cauchy sequence* if and only if $\lim\limits_{n\to\infty} \delta(T_n) = 0$, where $\delta(T_n)$ is the diameter of T_n in the real line $\mathbb{R}$.

A sequence $\langle x_n \rangle$ in $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) is a Cauchy sequence in the sense familiar from elementary analysis if and only if it is a Cauchy sequence according to definition (4) in the sense of the usual metric on $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).

Example 1 **A sequence $\langle x_n \rangle \subset \mathbb{R}$ is a Cauchy sequence in $\mathbb{R}$. Examine whether the sequence $\langle (x_n, \sin x_n) \rangle$ is a Cauchy sequence in $\mathbb{R}^2$.**

Solution: Let $x, y \in \mathbb{R}$, with $x \neq y$. By Lagrange mean value theorem,

$$\begin{aligned} & |\sin x - \sin y| = |x - y||\cos \xi|, \qquad \min\{x, y\} < \xi < \max\{x, y\} \\ \Rightarrow \quad & |\sin x - \sin y| \leq |x - y|, \quad \forall\ x, y \in \mathbb{R}. \end{aligned}$$

Since $\langle x_n \rangle$ is a Cauchy sequence in $\mathbb{R}$ for any $\varepsilon > 0\ \exists$ a natural number k such that $|x_m - x_n| < \varepsilon/\sqrt{2}\ \forall\ m, n \geq k$. Now

$$\begin{aligned} & \sqrt{(x_m - x_n)^2 + (\sin x_m - \sin x_n)^2} \\ \leq \quad & \sqrt{(x_m - x_n)^2 + (x_m - x_n)^2} \\ = \quad & \sqrt{2}|x_m - x_n| < \sqrt{2} \cdot \frac{\varepsilon}{\sqrt{2}} = \varepsilon; \forall m, n \geq k. \end{aligned}$$

Thus $\langle (x_n, \sin x_n) \rangle$ is a Cauchy sequence in $\mathbb{R}^2$.

EXAMPLE **2** **Let $X = C[0,1]$ be a metric space with the metric d_∞ defined by**

$$d_\infty(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|, \text{ for all } f,g \in X.$$

Prove that $\langle x_n \rangle$ in X given by $x_n(t) = \frac{nt}{n+t}, \forall t \in [0,1]$ is a Cauchy sequence.

Solution: For $m \geq n$, the function

$$g(t) = x_m(t) - x_n(t) = \frac{mt}{m+t} - \frac{nt}{n+t} = \frac{(m-n)t^2}{(m+t)(n+t)}$$

being continuous on $[0,1]$, assumes its maximum at some point $t_0 \in [0,1]$. So

$$\begin{aligned} d_\infty(x_m,x_n) &= \sup_{t\in[0,1]} |x_m(t) - x_m(t)| = \frac{(m-n)t_0^2}{(m+t_0)(n+t_0)} \\ &\leq \frac{t_0^2}{n+t_0} \leq \frac{1}{n} \to 0, \text{ for large } m,n. \end{aligned}$$

Moreover, the sequence $\langle x_n \rangle$ converges to some limit. Indeed, let $x(t) = t$, then

$$|x_n(t) - x(t)| = \Big|\frac{nt}{n+t} - t\Big| = \frac{t^2}{n+t} \leq \frac{1}{n} \to 0, \text{ as } n \to \infty.$$

Therefore, $\langle x_n \rangle$ converges to the limit x, where $x(t) = t$ for all $t \in [0,1]$.

THEOREM **8** **In a metric space every convergent sequence is a Cauchy sequence.**

Proof: Let (X,d) be a metric space. Suppose $\langle x_n \rangle$ be a convergent sequence in X converging to a. Then for any $\varepsilon > 0$, $\exists$ a natural number k such that $d(x_n,a) < \varepsilon/2$, $\forall\ n \geq k$. Now, by triangle inequality

$$\begin{aligned} d(x_m,x_n) &\leq d(x_m,a) + d(a,x_m) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall\ m,n \geq k. \end{aligned}$$

Thus $\langle x_n \rangle$ is a Cauchy Sequence.

RESULT **4** A Cauchy sequence may not be convergent as may be seen in the following examples:

(i) Let $X = \mathbb{Q}$ be the set of all rational numbers and $d(x,y) = |x-y|$, for all $x,y \in X$ be the usual metric on X. Consider the sequence $\langle x_n \rangle$ represented in decimal system, such that $x_n = 1.a_1a_2\cdots a_n$ is the largest rational number satisfying $x_n^2 < \sqrt{2}$. Then we have the following sequence of rational numbers

$$\begin{aligned} & x_1 = 1.4, x_2 = 1.41, x_3 = 1.414, x_4 = 1.4142, \cdots . \\ \therefore\quad & d(x_m,x_n) = |x_m - x_n| = 0.00\cdots 0a_{m+1}\cdots a_n < \frac{1}{10^m}; \forall n \geq m. \end{aligned}$$

Therefore, $d(x_m,x_n) \to 0$ as $m \to \infty$. Hence the sequence $\langle x_n \rangle$ is a Cauchy sequence and converges to the irrational number $\sqrt{2} \notin \mathbb{Q}$.

(ii) Let $X = (0, 1]$ be a metric space with the usual metric and x_n be the sequence in X where $x_n = \frac{1}{n}$ where $n \in \mathbb{N}$. Then $\langle x_n \rangle$ is a Cauchy sequence, since for each $\varepsilon > 0$

$$|x_m - x_n| = |\frac{1}{m} - \frac{1}{n}| < \varepsilon \qquad \forall\ m, n \geq [\frac{1}{\varepsilon}] + 1$$

But $\langle x_n \rangle$ does not converge to any point in X as 0 does not belongs to X. Since $\langle 1/n \rangle$ tends to 0 in $\mathbb{R}$ and 0 is the unique limit of the sequence $\langle 1/n \rangle$ in $\mathbb{R}$, no point of $(0, 1]$ can be the limit of $\langle 1/n \rangle$.

THEOREM **9 Let $\langle x_n \rangle$ be a Cauchy sequence in a metric space (X, d). Then $\langle x_n \rangle$ is convergent iff it has a convergent sub-sequence.**

Proof: If $\langle x_n \rangle$ is a convergent then, by Theorem 2 every sub-sequence of $\langle x_n \rangle$ is convergent.

Conversely: Suppose $\langle x_{n_k} \rangle$ ba a convergent sub-sequence of $\langle x_n \rangle$ such that $\langle x_{n_k} \rangle \to a$ as $k \to \infty$. Let $\varepsilon > 0$ be given. Since $\langle x_n \rangle$ is a Cauchy sequence $\exists$ a natural number N_1 such that

$$d(x_m, x_n) < \frac{\varepsilon}{2} \qquad \forall\ m, n \geq N_1$$

Let $N = \max\{N_1, n_{k_1}\}$ and k_2 be a natural number such that $n_{k_2} \geq N$. So

$$\begin{aligned} d(x_n, a) &\leq d(x_n, x_{n_{k_2}}) + d(x_{n_{k_2}}, a) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \forall n \geq N \\ \text{i.e.} \qquad d(x_n, a) &< \varepsilon \qquad \forall\ n \geq N. \end{aligned}$$

Hence the sequence $\langle x_n \rangle$ is a convergent.

EXAMPLE **3 If $\langle x_n \rangle$ and $\langle y_n \rangle$ are two Cauchy sequences in a metric space (X, d), Show that $\langle d(x_n, y_n) \rangle$ is an convergence sequence of reals.**

Solution: Since $\langle x_n \rangle$ and $\langle y_n \rangle$ are Cauchy sequence in (X, d) for $\varepsilon > 0$ $\exists$ natural number k_1, k_2 such that

$$d(x_m, x_n) < \frac{\varepsilon}{2}; \forall m, n \geq k_1 \text{ and } d(y_m, y_n) < \frac{\varepsilon}{2} \qquad \forall\ m, n \geq k_2.$$

Let $k = \max\{k_1, k_2\} \in \mathbb{N}$. Then Then

$$\begin{aligned} \big|d(x_m, y_m) - d(x_n, y_n)\big| &\leq d(x_m, x_n) + d(y_m, y_n) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall m, n \geq k. \end{aligned}$$

So, $\langle d(x_n, y_n) \rangle$ is a Cauchy Sequence of real numbers. Since every Cauchy sequence in $\mathbb{R}$ is convergent, $\langle d(x_n, y_n) \rangle$ is a convergent sequence in $\mathbb{R}$.

THEOREM **10 Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a uniformly continuous function. If $\langle x_n \rangle$ is a Cauchy sequence in X, then $\langle f(x_n) \rangle$ is a Cauchy sequence in Y.**

Proof: Since f is uniformly continuous, corresponding to an $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$d(x, x') < \delta; x, x' \in X \Rightarrow \rho\Big(f(x), f(x')\Big) < \varepsilon.$$

In particular, we have

$$d(x_m, x_n) < \delta \Rightarrow \rho\Big(f(x_m), f(x_n)\Big) < \varepsilon. \quad (5)$$

Since $\langle x_n \rangle$ is a Cauchy sequence in X, given $\delta > 0$, $\exists n_0 \in \mathbb{N}$, such that

$$d(x_m, x_n) < \delta, \quad \forall m, n \geq n_0. \quad (6)$$

Therefore, from Eqs. (5) and (6), it follows that

$$\rho\Big(f(x_m), f(x_n)\Big) < \varepsilon, \quad \forall m, n \geq n_0. \quad (7)$$

Hence $\langle f(x_n) \rangle$ is a Cauchy sequence in Y.

EXAMPLE 4 **Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be sequences in a metric space (X, d) such that $\langle y_n \rangle$ is Cauchy and $d(x_n, y_n) \to 0$ as $n \to \infty$. Prove that**

(i) $\langle x_n \rangle$ is a Cauchy sequence in X

(ii) $\langle x_n \rangle$ converges to $a \in X$ if and only if $\langle y_n \rangle$ converges to a.

Solution: (i) Let $\varepsilon > 0$ be chosen arbitrary. Since $\langle y_n \rangle$ is a Cauchy sequence, $\exists\ n_1 \in \mathbb{N}$ such that $d(y_m, y_n) < \frac{\varepsilon}{3}$; for all $m, n > n_1$.

Since $d(x_n, y_n) \to 0$ as $n \to \infty$, $\exists\ n_2 \in \mathbb{N}$ such that $n_2 > \frac{3}{\varepsilon}$, and

$$d(x_n, y_n) < \varepsilon/3; \text{ for all } n > n_2.$$

Let $n_0 = \max\{n_1, n_2\}$ $d(x_n, y_n) < \frac{\varepsilon}{3}$; for all $n > n_2$ by the triangle inequality, we have

$$\begin{aligned} d(x_n, y_n) &\leq d(x_m, y_m) + d(y_m, y_n) + d(y_n, x_n) \\ &< \frac{\varepsilon}{3} + d(y_m, y_n) + \frac{\varepsilon}{3} \ ; \ \text{ for all } n, m \geq n_2 \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \ ; \ \text{ for all } n, m > n_2 \end{aligned}$$

Thus $\langle x_n \rangle$ is a Cauchy Sequence.

(ii) By the triangle inequality, we have

$$\begin{aligned} & d(y_n, a) \leq d(y_n, x_n) + d(x_n, a) \\ & \lim_{n\to\infty} d(y_n, a) \leq \lim_{n\to\infty} d(y_n, x_n) + \lim_{n\to\infty} d(x_n, a) \\ & \leq 0 + 0 = 0. \end{aligned}$$

Thus $y_n \to a$ as $n \to \infty$.

2 Equivalence of Metrices

There are many metrices on a given set. For example, we have three different metrices on $\mathbb{R}^2$. Amoung these metrices, some of them have the same nature in relation to convergence of sequences and continuity of function. This leads to the notion of equivalent metrices on a set.

Definition 5 [Equivalent metric space]: Let d and d^* be two metrices on the set X such that for every $\langle x_n \rangle$ in X and $x \in X$

$$\lim_{n\to\infty} x_n = x \text{ in } (X,d) \text{ iff } \lim_{n\to\infty} x_n = x \text{ in } (X,d^*).$$

Then the two metrices d and d^* on the same underlying set X are said to be *equivalent*.

Using the above definition, we have the following theorem giving the condition for two metrices on a set X to be equivalent.

Theorem **11 (Necessary condition) Let d and d^* are two metrices on a nonempty set X and if there exists a constant K such that**

$$\frac{1}{K} d^*(x,y) \le d(x,y) \le K d^*(x,y); \quad \forall x,y \in X, \tag{8}$$

then the metrices d and d^* are equivalent.

Proof: Let $\langle x_n \rangle$ be a sequence in X such that $x_n \to x$ in (X,d). Using the left part of the inequality Eq. (8), we have

$$\frac{1}{K} d^*(x_n,n) \le d(x_n,x) \to 0 \text{ as } n \to \infty$$

shows that $x_n \to x$ in (X,d^*). Conversely, if $x_n \to x$ in (X,d^*), then the right part of the inequality (8), we have

$$d(x_n,n) \le K\ d^*(x_n,x) \to 0 \text{ as } n \to \infty$$

shows that $x_n \to x$ in (X,d). Hence, from the definition d and d^* on X are equivalent.

Example **5 Prove that the metric space (X,d) and (X,d^*), where $d^*(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$ are equivalent.**

Solution: Let $x_n \to x$ in the metric d so that $d(x_n,x) \to 0$ as $n \to \infty$. Then obviously, $d^*(x_n,x) \to 0$ as $n \to \infty$ so that $x_n \to x$ in the metric d^*.

Conversely, let $x_n \to x$ in the metric d^* so that $d^*(x_n,x) \to 0$ as $n \to \infty$. So corresponding to an $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$\begin{aligned} & d^*(x_n,x) < \varepsilon, \text{ for } n \ge n_0 \\ \Rightarrow \quad & \frac{d(x_n,x)}{1+d(x_n,x)} < \varepsilon, \text{ for } n \ge n_0 \\ \Rightarrow \quad & d(x_n,x) < \frac{\varepsilon}{1-\varepsilon}, \text{ for } n \ge n_0. \end{aligned}$$

Choosing $\varepsilon < 1$, $d^*(x_n,x) \to 0$ as $n \to \infty$ implies $d(x_n,x) \to 0$ as $n \to \infty$. Therefore two metrices d and d^* are equivalent. On the other hand, we observe that there is no constant $K > 0$ such that $d(x,y)/K \le d^*(x,y); \forall x,y \in \mathbb{R}$.

Example 6 **Let c be the set of all convergent sequences with the following metrices d and d^*. If $x = \langle x_n \rangle$ and $y = \langle y_n \rangle$ are in c, let**

$$d(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n|; \quad d^*(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

Prove that d and d^* on c are not equivalent.

Solution: To prove that the two metrices d and d^* on c are not equivalent, we produce a sequence which converges to a limit in one metric but does not converge in the other metric. Consider the sequence $\langle e_n \rangle$, in c whose respective terms are sequences

$$e_1 = (1, 0, 0, \cdots), e_2 = (0, 1, 0, \cdots), \cdots, e_n = (0, 0, 0, \cdots, 0, 1, \cdots),$$

where 1 is in the n^{th} place. Let $e_0 = (0, 0, 0, \cdots, 0)$. In the metric space c of convergent sequences

$$d(e_n, e_0) = 1; \quad d^*(e_n, e_0) = \frac{1}{2^n} \frac{1}{1+1} = \frac{1}{2^{n+1}} \to 0 \text{ as } n \to \infty.$$

Thus the sequence $\langle e_n \rangle$ converges with respect to the metric d but not with respect to the metric d^*. Hence the metrices d and d^* are not equivalent.

Example 7 **Let $\mathcal{C}[0, 1]$ be a space of all continuous linear functionals defined on $[0, 1]$. Consider the metrices d and d^* defined on $\mathcal{C}[0, 1]$ by**

$$d(f, g) = \sup_{0 \le t \le 1} |f(t) - g(t)|; \quad d^*(f, g) = \int_0^1 |f(t) - g(t)| dt,$$

for all $f, g \in \mathcal{C}[0, 1]$. Prove that they are not equivalent.

Solution: To prove that the two metrices d and d^* on $\mathcal{C}[0, 1]$ are not equivalent, we produce a sequence which converges to a limit in one metric but does not converge in the other metric. Consider the sequence $\langle f_n(x) \rangle$ in $\mathcal{C}[0, 1]$, defined by

$$f_n(x) = x^n; \quad \text{for all } x \in [0, 1].$$

Also, let $f(x) = 0$ for all $x \in [0, 1]$. Then

$$d(f_n, f) = \sup_{0 \le t \le 1} |f_n(t) - f(t)| = 1, \quad \forall n \in \mathbb{N},$$
$$d^*(f_n, f) = \int_0^1 t^n \, dt = \frac{1}{n+1} \to 0 \text{ as } n \to \infty.$$

Therefore, $\lim_{n \to \infty} d(f_n, f) \neq 0$. Thus the sequence $\langle f_n(x) \rangle$ converges to f with respect to the metric d^* but not with respect to d. Therefore the metrices d and d^* are not equivalent.

However, for any sequence $\langle f_n(x) \rangle$ in $\mathcal{C}[0, 1]$ and any $f \in \mathcal{C}[0, 1]$, $d(f_n, f) \to 0$ as $n \to \infty$, it follows that $d^*(f_n, f) \to 0$ as $n \to \infty$. Observe that

$$d^*(f_n, f) = \int_0^1 |f_n(t) - f(t)| dt \le \sup_{0 \le t \le 1} |f(t) - g(t)| = d(f_n, f).$$

3 Complete Metric Space

We now turn to some aspects of convergence in metric space. In our study of Cauchy sequences in $\mathbb{R}$, we have noted that every Cauchy sequence converges in the real line. In the general case of metric spaces, this need not happen by following examples

(i) Let us consider $X = (0, 1)$ as a subspace of $\mathbb{R}$ with usual metric. Then X is a metric space in its own right. Then $\langle 1/n \rangle$ is a Cauchy sequence in X but not convergent in X. Hence $(0, 1)$ is not complete with respect this metric.

(ii) In the metric space $(\mathbb{Q}, d)$, the sequence defined by $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$, whose elements become closer and closer to each other without converging to an element of $\mathbb{Q}$, even though it is a Cauchy sequence.

That is there are metric spaces in which not all Cauchy sequences of points converge to a point of the metric space. Hence, it is worthwhile to single out those spaces in which Cauchy sequence converges to a point of the metric space.

Definition 6 A metric space (X, d) is said to be *complete* if and only if, every Cauchy sequence of elements of X converges to some element of X in the space.

(i) The definition of complete metric space at once suggests that, if $d(x_n, x_m) \longrightarrow 0$ as $m, n \to \infty$, then $\exists$ a $x_0 \in X$ such that $d(x_n, x_0) \longrightarrow 0$ as $n \to \infty$.

(ii) A metric space (X, d) is complete if and only if every Cauchy sequence in X has a convergent subsequence.

Many metric spaces, with or without ordering, have this property. We define completeness as universal closure. For example,

(i) the usual metric $\mathbb{R}_u$ is a complete.

(ii) the usual metric $\mathbb{C}_u$ is a complete.

(iii) Every finite-dimensional normed linear space is complete.

Below, we shall give several examples of complete metric space.

EXAMPLE 8 **The set of integers $\mathbb{Z}$ with the usual metric is a complete metric space**.

Solution: Let $\langle x_n \rangle$ be a Cauchy sequence of integers, that is, each term of the sequence belongs to $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$. Then the sequence must be of the form $\{x_1, x_2, x_3, \cdots, x_n, x, x, x, \cdots\}$. Indeed, if we choose $\varepsilon = 1/2$, then

$$x_n, x_m \in \mathbb{Z} \text{ and } |x_n - x_m| < 1/2 \Rightarrow x_n = x_m.$$

Hence the sequence $\{x_1, x_2, x_3, \cdots, x_n, x, x, x, \cdots\}$ converges to x.

EXAMPLE 9 **Any set X with discrete metric forms a complete metric space.**

Solution: Let (X, d) be a discrete metric space and $\langle x_n \rangle$ be a Cauchy sequence in (X, d). Thus $d(x_m, x_n) = 0$ if $x_m = x_n$ and $d(x_n, x_m) = 1$ if $x_m \neq x_n$. Then for $\varepsilon = 1/2, \exists$ a natural number k such that

$$\begin{aligned} & d(x_m, x_n) < 1/2 \quad \forall\, m, n \geq k \\ \Rightarrow \quad & d(x_m, x_n) = 0 \quad \forall\, m, n \geq k, \quad \text{since } d \text{ is discrete metric} \\ \Rightarrow \quad & x_k = x_{k+1} = x_{k+2} = \cdots = x \quad \text{(say)} \end{aligned}$$

$\therefore d(x_n, x) = 0, \forall n \geq k$ i.e., $\langle x_n \rangle$ is of the form $\{x_1, x_2, \cdots, x_k, x, x, \cdots\}$. Hence $\langle x_n \rangle$ converges to x, i.e., every Cauchy sequence is convergent in a discrete metric space. Thus the discrete metric space (X, d) is a complete. From this we see that a sequence in a discrete metric space is Cauchy if and only if it is eventually constant.

In particular,, the set $(0, 1)$ with the discrete metric is a complete metric space.

Example **10 A Euclidian n-space $\mathbb{R}^n$ is a complete metric space with respect to usual metric.**

Solution: We know $\mathbb{R}^n$ is a metric space with respect to the metric d where, $d(x, y) = \left\{\sum_{i=1}^{n} (a_i - b_i)^2\right\}^{1/2}$ for all $x = (a_1, a_2, \cdots, a_n)$; $y = (b_1, b_2, \cdots, b_n)$ and $x, y \in \mathbb{R}^n$. Let us suppose the coordinate sequences $\langle x_m \rangle$ be Cauchy in $\mathbb{R}^n$, where $x_m = \left(a_1^{(m)}, a_2^{(m)}, \cdots, a_n^{(m)}\right)$ for $m = 1, 2, 3, \cdots$.

$\therefore$ For each $\varepsilon > 0 \; \exists$ a $k \in \mathbb{N}$ such that

$$\begin{aligned} & d(x_m, x_r) < \frac{\varepsilon}{2} \quad \forall m, r \geq k \\ \text{i.e.,} \quad & \left\{\sum_{i=1}^{n} (a_i^{(m)} - b_i^{(m)})^2\right\}^{\frac{1}{2}} < \frac{\varepsilon}{2} \quad \forall\, m, r \geq k \\ \Rightarrow \quad & \sum_{i=1}^{n} \left\{a_i^{(m)} - b_i^{(m)}\right\}^2 < \left(\frac{\varepsilon}{2}\right)^2 \quad \forall\, m, r \geq k. \end{aligned}$$

Since each term in the above inequality are positive so

$$\begin{aligned} & \left\{a_i^{(m)} - b_i^{(m)}\right\}^2 < \left(\frac{\varepsilon}{2}\right)^2 \quad \forall\, m, r \geq k \;\text{ for } i = 1, 2, 3, \cdots, n \\ \Rightarrow \quad & \left|a_i^{(m)} - b_i^{(m)}\right| < \frac{\varepsilon}{2} \quad \forall\, m, r \geq k \;\text{ for } i = 1, 2, 3, \cdots, n. \end{aligned}$$

$\therefore \left\{a_i^{(m)}\right\}$ or $\left\{a_i^{(1)}, a_i^{(2)}, \cdots\right\}$ is a Cauchy sequence in $\mathbb{R}$ for $i = 1, 2, \cdots, n$. Since $\mathbb{R}$ is complete, $\langle a_i^{(m)} \rangle$ converges to a_i (say) in $\mathbb{R}$ for $i = 1, 2, \cdots, n$. So,

$$\lim_{r \to \infty} a_i^{(r)} = a_i \quad \text{for } i = 1, 2, \cdots, n$$

Let $x = \left(a_1, a_2, \cdots, a_n\right)$, then $x \in \mathbb{R}^n$. We now prove that $\langle x_m \rangle$ converges to x.

$$\begin{aligned} \therefore \quad \left\{d(x_m, x)\right\}^2 &= \sum_{i=1}^{n} \left\{a_i^{(m)} - a_i\right\}^2 = \lim_{r \to \infty} \sum_{i=1}^{n} \left\{a_i^{(m)} - a_i^{(r)}\right\}^2 \\ &= \left(\frac{\varepsilon}{2}\right)^2 \quad \forall\, m \geq k \\ \therefore \quad d(x_m, x) &\leq \frac{\varepsilon}{2} < \varepsilon \quad \forall\, m \geq k \end{aligned}$$

Hence the Cauchy sequence $\langle x_m \rangle$ converges to $x \in \mathbb{R}^n$. Thus $\mathbb{R}^n$ is a complete metric space.

Result **5** Since the completeness of the metric is not destroyed by replacing an equivalent metric, the above is true for any one of the three equivalent metrices for $\mathbb{R}^n$.

Theorem **12 The space $\mathcal{C}[a,b]$ of all continuous real-valued functions defined on $[a,b]$ with the metric $d_\infty(f,g) = \max\limits_{t\in[a,b]} |f(t) - g(t)|$ is a complete metric space.**

Proof: Let $\langle f_n \rangle$ be a Cauchy sequence of elements in $\mathcal{C}[a,b]$. Then for each $\varepsilon > 0$, there exists a $k \in \mathbb{N}$ such that

$$\begin{aligned} & d_\infty(f_m, f_n) < \varepsilon; \quad \forall\, m,n \geq k \\ \Rightarrow \quad & \sup_{t\in[a,b]} \left|f_m(t) - f_n(t)\right| < \varepsilon; \forall\, m,n \geq k \\ \Rightarrow \quad & \left|f_m(t) - f_n(t)\right| < \varepsilon; \forall\, m,n \geq k \text{ and for } t \in [a,b]. \end{aligned}$$

$\therefore$ By Cauchy's general principle of convergence, $\langle f_n \rangle$ converges uniformly. Let f be the limit function. Since each $\langle f_n \rangle$ is a continuous function on $[a,b]$ and $\langle f_n \rangle$ converges uniformly to f on $[a,b]$, f is a continuous function on $[a,b]$ i.e., $f \in \mathcal{C}[a,b]$. Again since $\langle f_n \rangle$ converges uniformly to f on $[a,b]$, for any $\varepsilon > 0$, $\exists$ a $k \in \mathbb{N}$ such that

$$\begin{aligned} & \left|f_n(t) - f(t)\right| < \frac{\varepsilon}{2} \qquad \forall\, n \geq k,\ \forall\, t \in [a,b] \\ \Rightarrow \quad & \sup_{t\in[a,b]} \left|f_n(t) - f(t)\right| \leq \frac{\varepsilon}{2} \qquad \forall\, n \geq k \\ \Rightarrow \quad & d(f_n, f) \leq \frac{\varepsilon}{2} < \varepsilon \qquad \forall\, n \geq k. \end{aligned}$$

Thus $\langle x_n \rangle$ converges to x on $\mathcal{C}[a,b]$. Hence $\mathcal{C}[a,b]$ is a complete metric metric space and it is defined as the *space of continuous functions on I*.

Example **11 The Hilbert space $(l_p, d), p \geq 1$ is complete.**

Solution: The Hilbert space $l_p (1 \leq p < \infty)$ of all sequences $\langle x_n \rangle$ of real or complex numbers such that $\sum\limits_{k=1}^{\infty} |x_k|^p < \infty$ with the metric given by

$$d(x,y) = \left[\sum_{i=1}^{\infty} \left|\xi_i - \eta_i\right|^p\right]^{1/p}; x = \{\xi_i\}, y = \{\eta_i\}.$$

Let $\langle t_n \rangle$, where $t_n = \langle x_i^n \rangle_{i=1}^{\infty}$, be a Cauchy sequence in l_p. Let $\varepsilon > 0$ be a real number. Then there exists an $n_0 \in \mathbb{N}$ such that

$$d(t_n, t_m) < \varepsilon, \text{ for all } n,m \geq n_0. \tag{9}$$

This shows that $|x_i^n - x_i^m| < \varepsilon$ for all $n,m \geq n_0$ and consequently $\langle x_i^n \rangle_{i=1}^{\infty}$ is Cauchy in $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Since these spaces are complete, $\langle x_i^n \rangle_{i=1}^{\infty}$ converges to a point $x_i \in \mathbb{K}$. Also for each $k \in \mathbb{N}$, the statement (9) gives

$$\begin{aligned} & \sum_{i=1}^{k} |x_i^n - x_i^m| \leq \varepsilon^p \text{ for all } n,m \geq n_0 \\ \Rightarrow \quad & \sum_{i=1}^{k} |x_i^n - x_i| \leq \varepsilon^p \text{ for all } n \geq n_0, \text{ as } m \to \infty. \end{aligned} \tag{10}$$

We need to prove that $t = \langle x_1, x_2, \cdots \rangle$ is in l_p. The inequalities (10) above and Minkowski's inequality show that

$$\begin{aligned}
\left[\sum_{i=1}^{k} \left|x_i\right|^p\right]^{1/p} &\leq \left[\sum_{i=1}^{k} \left\{\left|x_i^{n_0} - x_i\right|^p + \left|x_i^{n_0}\right|^p\right\}\right]^{1/p} \\
&\leq \left[\sum_{i=1}^{k} \left|x_i^{n_0} - x_i\right|^p\right]^{1/p} + \left[\sum_{i=1}^{k} \left|x_i^{n_0}\right|^p\right]^{1/p} \\
&\leq \varepsilon + \left[\sum_{i=1}^{k} \left|x_i^{n_0}\right|^p\right]^{1/p}.
\end{aligned}$$

Since $t_{n_0} = \langle x_i^{n_0} \rangle_{i=1}^{\infty}$ is in l_p, the above inequality shows that $\left\langle \left[\sum_{i=1}^{k} \left|x_i\right|^p\right]^{1/p} \right\rangle$ is bounded and it is monotonically increasing, and hence the series $\sum_{i=1}^{k} \left|x_i\right|^p$ is convergent. Thus, t is in l_p. Also, it is obvious from (10) that $\langle t_n \rangle$ converges to t, and the completeness of l_p-space is now established.

Example 12 **Let (X, d) be a metric space and d^* be the standard bounded metric on X defined by $d^*(x, y) = \min\{1, d(x, y)\}$. Show that (X, d) is complete if and only if (X, d^*) is complete.**

Solution: First assume that (X, d) is complete and $\langle x_n \rangle$ is a Cauchy sequence in (X, d^*). Let $\varepsilon > 0$ be a real number. Therefore, for $0 < \varepsilon' < \min\{\varepsilon, 1\}$, there exists a $p \in \mathbb{N}$ such that

$$\begin{aligned}
& d^*(x_n, x_m) < \varepsilon' \quad \text{for all } n, m \geq p \\
\Rightarrow \quad & d(x_n, x_m) < \varepsilon' < \varepsilon \quad \text{for all } n, m \geq p. \qquad (11)
\end{aligned}$$

Thus, $\langle x_n \rangle$ is a Cauchy sequence in (X, d). Since (X, d) is complete, $\langle x_n \rangle$ converges to a point $x \in X$. Letting m tend to infinity in Eq. (11), it follows that

$$d(x_n, x) < \varepsilon \quad \text{for all } n \geq p.$$

Now the completeness $d(X, d^*)$ follows from the inequalities

$$d^*(x_n, x_m) \leq d(x_n, x) < \varepsilon \quad \text{for all } n \geq p.$$

Conversely, let $\langle x_n \rangle$ is a Cauchy sequence in (X, d). Let $\varepsilon > 0$ be a real number, and $0 < \varepsilon' < \min\{\varepsilon, 1\}$. Then there exists a $p \in \mathbb{N}$ such that

$$\begin{aligned}
& d(x_n, x_m) < \varepsilon' \quad \text{for all } n, m \geq p \\
\Rightarrow \quad & d^*(x_n, x_m) < \varepsilon' < \varepsilon \quad \text{for all } n, m \geq p. \qquad (12)
\end{aligned}$$

Thus, $\langle x_n \rangle$ is a Cauchy sequence in (X, d^*). Since (X, d^*) is complete, $\langle x_n \rangle$ converges to a point $x \in X$. Letting m tend to infinity in Eq. (12), it follows that

$$\begin{aligned}
& d^*(x_n, x) < \varepsilon' < \varepsilon \quad \text{for all } n \geq p \\
& d(x_n, x) \leq \varepsilon \quad \text{for all } n \geq p
\end{aligned}$$

and the complete is complete.

Example **13** **Let X be the set of all convergent real (or complex) sequences with the function $d : X \times X \to \mathbb{R}$ is defined by**

$$d(x, y) = \sup_n \Big\{ |x_n - y_n| \Big\}; \quad x = \{x_n\}, y = \{y_n\} \in X.$$

Prove that (X, d) is a complete metric space.

Solution: In Example (**??**), we see that (X, d) is a metric space. Let $\langle t_n \rangle$, where $t_n = \langle x_{n1}, x_{n2}, \cdots, x_{nk}, \cdots \rangle$ for all $n \in \mathbb{N}$ is an element of X be a Cauchy sequence in X. Let $\varepsilon > 0$ be a real number. Then $\exists$ a $p \in \mathbb{N}$ such that

$$\begin{aligned} & d(t_n, t_m) < \varepsilon; \quad \forall n, m \geq p \\ \Rightarrow \quad & \sup_{k \in \mathbb{N}} |x_{nk} - x_{mk}| < \varepsilon; \quad \forall n, m \geq p. \end{aligned} \tag{13}$$

Therefore, for each $k \in \mathbb{N}$, the sequence $\langle x_{nk} \rangle_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$(or $\mathbb{C}$), and using the completeness of $\mathbb{R}$(or $\mathbb{C}$), the sequence $\langle x_{nk} \rangle_{n=1}^{\infty}$ converges to a point $x_k \in \mathbb{R}$ (or $\mathbb{C}$). Thus we get a sequence $t = \langle x_1, x_2, \cdots \rangle$. When we get $m \to \infty$ in Eq. (13), it follows that

$$|x_{nk} - x_k| < \varepsilon, \text{ for all } n \geq p. \tag{14}$$

Now, by inequalities

$$\begin{aligned} |x_k| &\leq |x_k - x_{pk}| + |x_{pk}|, \text{ for all } k \in \mathbb{N} \\ \Rightarrow \sup_{k \in \mathbb{N}} |x_k| &\leq \sup_{k \in \mathbb{N}} |x_k - x_{pk}| + \sup_{k \in \mathbb{N}} |x_{pk}| \\ &\leq \varepsilon + \sup_{k \in \mathbb{N}} \{|x_{pk}|\}. \end{aligned}$$

Since t_p is bounded, it follows that $t \in X$. Taking supremum over k in Eq. (14), shows that $\langle t_n \rangle$ converges to t, and (X, d) is complete.

Example **14** **Let $\mathcal{D}[a, b]$ denote the set of all functions f on $[a, b]$ which have continuous derivatives at all points of $I = [a, b]$. For $f, g \in \mathcal{D}[a, b]$, define**

$$d(f, g) = |f(a) - g(b)| + \sup\{|f'(x) - g'(x)| : x \in I\}.$$

Show that d is a metric for $\mathcal{D}[a, b]$ and that the space $(\mathcal{D}[a, b], d)$ is complete.

Solution: Let $\langle f_n \rangle$ be a Cauchy sequence in $\mathcal{D}[a, b]$. Then for a given $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$\begin{aligned} & d(f_m, f_n) < \varepsilon; \quad m, n \geq m_0 \\ \Rightarrow \quad & |f_m(a) - f_n(b)| + \sup\{|f_m'(x) - f_n'(x)| : x \in I\} < \varepsilon, \\ \Rightarrow \quad & |f_m'(x) - f_n'(x)| < \varepsilon; \quad m, n \geq m_0 \text{ and all } x \in I. \end{aligned}$$

Hence $\langle f_n' \rangle$ is an uniformly convergent sequence of continuous functions and must therefore converge to the continuous function ϕ, say. Then by a well known theorem on analysis, the sequence $\langle f_n \rangle$ converges uniformly to f such that

$$\lim_{n \to \infty} f_n'(x) = f'(x) = \phi(x); \quad a \leq x \leq b.$$

Thus $\langle f_n \rangle$ converges to a continuously differentiable function f so that $f \in \mathcal{D}[a, b]]$. It follows that the space, $\mathcal{D}[a, b]$, is complete.

Theorem **13 [Cantor intersection theorem] : Let (X, d) be a metric space. A necessary and sufficient condition that the metric space X be complete is that every nested sequence $\langle F_n \rangle$ of non-empty closed subsets of X with $\delta(F_n) \to 0$ as $n \to \infty$ be such that $F = \bigcap_{i=1}^{\infty} F_i$ contains exactly one point.**

Proof: Let the metric space (X, d) be complete. Consider a sequences of closed intervals $\langle F_n \rangle$ such that

$$F_1 \supset F_2 \supset \cdots \text{ and } \delta(F_n) \to 0 \text{ as } n \to \infty.$$

Let us construct a sequence $\langle a_n \rangle$ in a_0 by selecting a point $a_n \in F_n \ \forall\ n \in \mathbb{N}$ (since each F_n is non-empty). Since $\delta(F_n) \to 0$, for every $\varepsilon > 0$, $\exists$ a positive integer $m_0 \in \mathbb{N}$ such that $\delta(F_{m_0}) < \varepsilon$. Again since sequence $\langle F_n \rangle$ is nested (monotone decreasing), for every positive integer, we have

$$\begin{aligned} n, n+p \geq m_0 \Rightarrow \quad & F_{n+p}, F_n \subset F_{m_0} \Rightarrow a_n, a_{n+p} \in F_{m_0} \\ \Rightarrow \quad & d(a_n, a_{n+p}) \leq \delta(F_{m_0}) < \varepsilon \\ \Rightarrow \quad & d(a_n, a_{n+p}) \to 0 \text{ as } n \to \infty. \end{aligned}$$

$\therefore \langle a_n \rangle$ is a Cauchy sequence in (X, d). Since the metric space (X, d) is complete, $a_n \to a_0$ for some $a_0 \in X$.

We now prove that $a_0 \in \bigcap_{n=1}^{\infty} F_n$. Now the sub-sequence $\left\langle a_n, a_{n+1}, a_{n+2}, \cdots \right\rangle$ of $\langle a_n \rangle$ is contained in F_n and still converges to a_0. But F_n being a closed sub-space of (X, d), $a_0 \in F_n$. Since this is true for each $n \in \mathbb{N}$, $a_0 \in \bigcap_{n=1}^{\infty} F_n$.

Suppose $a_0^* \in \bigcap_{n=1}^{\infty} F_n$. Then $a_0, a_0^* \in F_n$ for each $n \in \mathbb{N}$, so

$$\begin{aligned} & 0 \leq d(a_0, a_0^*) \leq \delta(F_n) \to 0 \text{ as } n \to \infty \\ \therefore \quad & d(a_0, a_0^*) = 0 \Rightarrow a_0 = a_0^*. \end{aligned}$$

Conversely, let F consists of a single point for every nested sequence $\langle F_n \rangle$ of non empty closed subsets F_n of X such that $\delta(F_n) \to 0$. We are to show that X is complete.

Let $\langle a_n \rangle$ be any Cauchy sequence in X. Let $H_n = \left\langle x_n, x_{n+1}, x_{n+2}, \cdots \right\rangle$. Since $\langle x_n \rangle$ is Cauchy, for a given $\varepsilon > 0$, $\exists$ a $m_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon \text{ for } n, m \geq m_0.$$

It follows that $\delta(H_n) < \varepsilon$ for $n \geq m_0$ and consequently, $\delta(H_n) \to 0$ as $n \to \infty$. Also,

$$H_1 \supset H_2 \supset \cdots \Rightarrow \bar{H}_1 \supset \bar{H}_2 \supset \cdots$$

and $\delta(H_n) = \delta(\bar{H}_n)$. Hence

$$\delta(\bar{H}_n) < \varepsilon \text{ for } n \geq m_0 \Rightarrow \delta(\bar{H}_n) \to 0 \text{ as } n \to \infty.$$

Hence $\langle \bar{H}_n \rangle$ is a nested sequence of closed subsets of non empty sets in X whose diameters tend to zero. Then by hypothesis, $\exists$ a unique point x_0 such that $x_0 \in \bigcap_{n=1}^{\infty} \bar{H}_n$.

We claim that the Cauchy sequence $\langle x_n \rangle$ converges to x_0. Since $\delta(\bar{H}_n) \to 0$ for a given $\varepsilon > 0$, $\exists$ a $m_0 \in \mathbb{N}$ such that $\delta(\bar{H}_{m_0}) < \varepsilon$ and so

$$x_n, x_0 \in \bar{H}_{m_0} \Rightarrow d(x_n, x_0) < \varepsilon \text{ for } n \geq m_0.$$

This implies that $\langle x_n \rangle$ converges to x_0. Thus every Cauchy sequence in X converges to a point in X. Hence X is complete.

EXAMPLE **15 The real line $\mathbb{R}$ is a complete metric space, with respect to usual metric for $\mathbb{R}$.**

Solution: Let $\langle x_n \rangle$ be a Cauchy sequence of real numbers $\mathbb{R}$. We define the sequence $\langle n_k \rangle$ of positive integers by induction as follows: n_{k+1} is the smallest integer greater than n_k such that

$$\left| x_n - x_m \right| \leq 1/2^{k+1}; \;\; n, m \geq n_k.$$

Let I_k be the closed interval $\left[x_{n_k} - 2^{-k}, x_{n_k} + 2^{-k} \right]$. Then we have, $I_{k+1} \subset I_k$. For we have

$$\left| x_{n_k} - x_{n_{k+1}} \right| \leq 1/2^{k+1}.$$

Also, the length of $I_k \to 0$ as $k \to \infty$. Hence by nested interval theorem, $\bigcap\limits_{k=1}^{\infty} I_k$ contains exactly one point, say $a \in \mathbb{R}$. Thus $a \in I_k$ for all $k \in \mathbb{N}$ so that

$$\left| a - x_{n_k} \right| \leq 1/2^k; \;\; \forall k \in \mathbb{N}.$$

Again for $n \geq n_k$, we have

$$\left| x_{n_k} - x_n \right| \leq 1/2^{k+1} < 1/2^k.$$

Hence for all $n \geq n_k$, we have

$$\begin{aligned} |a - x_n| &= \left| a - x_{n_k} + x_{n_k} - x_n \right| \\ &\leq \left| a - x_{n_k} \right| + \left| x_{n_k} - x_n \right| \\ &< \frac{1}{2^k} + \frac{1}{2^k} = \frac{1}{2^{k-1}}. \end{aligned}$$

$\therefore \lim\limits_{n \to \infty} x_n = a$. Thus every Cauchy sequence in $\mathbb{R}$ converges to a point in $\mathbb{R}$ and consequently $\mathbb{R}$ is complete. The Euclidean metric on $\mathbb{R}$ is derived from the absolute-value function, which in turn depends on the ordering of $\mathbb{R}$. The fact that the ordering of $\mathbb{R}$ is complete is crucial in establishing that $\mathbb{R}$ is universally closed.

EXAMPLE **16 Show by an example that is (X, p) is not complete then $\bigcap\limits_{n=1}^{\infty} F_n$ may be empty where $\{F_n\}$ is a sequence of non-empty closed sets in X with $d(F_n) \to 0$ as $n \to \infty$.**

Solution: Let $X = (0,2]$. Then with respect to usual metric X is not a complete metric space, Since $\{\frac{1}{n}\}$ is a Cauchy sequence in X but it is not convergent in X. Let $F_n = (0, \frac{1}{n}]$

Then $\langle F_n \rangle$ is a monotonically decreasing sequence of non-empty closed set with $\delta(F_n) = \frac{1}{n} \to 0$ as $n \to \infty$. But $\bigcap\limits_{n=1}^{\infty} F_n = \phi$.

THEOREM **14** **Let (X,d) be a metric space. If for any monotonically decreasing sequence $\langle F_n \rangle$ of non-empty closed sets-with $\delta(F_n) \to 0$ as $n \to \infty$ imply $\bigcap\limits_{n=1}^{\infty} F_n$ contains exactly one point. Then (X,d) is complete.**

Proof: Let $\{x_n\}$ be a Cauchy sequence in (X,d). Suppose $G_n = \left\{x_n, x_{n+1}, x_{n+2}, \cdots\right\}$, then

$$G_1 \supseteq G_2 \supseteq G_2 \supseteq G_3 \supseteq \cdots \Rightarrow \overline{G_1} \supseteq \overline{G_2} \supseteq \overline{G_3} \supseteq \cdots.$$

Suppose $F_n = \overline{G_n}, \forall\ n$. Since closure of a set is closed set, F_n is a closed set for all n. So, $\langle F_n \rangle$ is a monotonically decreasing sequence of non-empty closed set. We now prove that $d(F_n) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ be given. Since $\langle x_n \rangle$ is a Cauchy sequence, for the above chosen ε $\exists$ a natural number k such that

$$\begin{aligned} & d(x_m, x_n) < \frac{\varepsilon}{2}; \qquad \forall\ m,n \geq k \\ \Rightarrow\ & \sup\{d(x_m,x_n)\ :\ m,n \geq k\} \ \leq\ \frac{\varepsilon}{2} \\ \Rightarrow\ & \delta(G_k) \leq \frac{\varepsilon}{2} < \varepsilon \Rightarrow \delta(\overline{G_k}) \leq \varepsilon, \text{ as } \ \delta(A) = \delta(\overline{A}) \\ \text{i.e.,}\ & \delta(F_k) < \varepsilon. \end{aligned}$$

Since $\langle F_n \rangle$ is monotonically decreasing, $\delta(F_n) < \varepsilon \forall\ n \geq k$. This implies $\delta(F_n) \to 0$ as $n \to \infty$. So by given condition $\exists$ a point $x \in X$ such that $x \in \bigcap\limits_{n=1}^{\infty} F_n$, i.e., $x \in F_n; \forall\ n$. We shall prove $\langle x_n \rangle$ converges to x.

Since $\delta(F_n) \to 0$ as $n \to \infty$ for any given $\varepsilon > 0$, $\exists$ a natural number m_0 such that $\delta(F_n) < \varepsilon, \forall\ n \geq m_0$. In particular, $\delta(F_{m_0}) < \varepsilon$, so

$$d(x_n, x) < \delta(F_{m_0}) < \varepsilon, \forall\ n \geq m_0$$
$$\text{as } \ x_n \in F_{m_0}, \ \ \forall\ n \geq m_0 \ \text{ and } \ x \in F_{m_0}.$$

Thus $\{x_n\}$ converges to x. Hence (X,d) is complete.

EXAMPLE **17** **Let (X,d_1) and (Y,d_2) be two complete metric spaces. Prove that the product space $Z = X \times Y$ with metric**

$$d(x,y) = \sqrt{d_1^2(x_1,y_1) + d_2^2(x_2,y_2)}$$

is complete, where $x = (x_1,x_2)$ and $y = (y_1,y_2)$.

Solution: From Example **??**, we see that d is a metric for Z. Let $\langle z_n \rangle$ be a Cauchy sequence in Z. Then for given $\varepsilon > 0$, $\exists$ a $n_0(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} & d(z_m, z_n) < \varepsilon, \ \text{whenever } m,n \geq n_0(\varepsilon) \\ \Rightarrow\ & d^2(z_m,z_n) < \varepsilon^2, \ \text{whenever } m,n \geq n_0(\varepsilon) \\ \Rightarrow\ & d_1^2(x_m,x_n) + d_2^2(y_m,y_n) < \varepsilon^2, \ \text{whenever } m,n \geq n_0(\varepsilon) \\ \Rightarrow\ & d_1^2(x_m,x_n) < \varepsilon^2 \text{ and } d_2^2(y_m,y_n) < \varepsilon^2, \ \text{whenever } m,n \geq n_0(\varepsilon) \\ \Rightarrow\ & d_1(x_m,x_n) < \varepsilon \text{ and } d_2(y_m,y_n) < \varepsilon, \ \text{whenever } m,n \geq n_0(\varepsilon). \end{aligned}$$

It follows that $\langle x_n \rangle$ and $\langle y_n \rangle$ are Cauchy sequence in the space X and Y respectively. Since these spaces are complete, the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ converge respectively to points $x \in X$ and $y \in Y$. It follows that $\langle z_n \rangle$ converges to $z = (x, y) \in Z$ and consequently the product metric space $Z = X \times Y$ is complete.

Result **6** The Cantor's intersection theorem may not valid if any any of the following two conditions is not satisfied:

(i) $\langle F_n \rangle$ is a sequence of closed sets

(ii) $\delta(F_n) \to 0$ as $n \to \infty$.

The following examples cite that no condition in the Cantor's intersection theorem can be dropped without losing the conclusion of the theorem.

(i) Let $X = \mathbb{R}$ be a complete metric space. Consider the sequence of open intervals $F_n = \left(0, \frac{1}{n}\right)$, for all $n \in \mathbb{N}$. Obviously

(a) $F_1 \supset F_2 \supset F_3 \supset \cdots$, i.e., $\langle F_n \rangle$ is a nest of open intervals

(b) $\delta(F_n) = \left|\frac{1}{n} - 0\right| = \frac{1}{n} \to 0$ as $n \to \infty$.

But the intersection $\bigcap_{n=1}^{\infty} F_n = \phi$.

(ii) In the real line $\mathbb{R}$, let us consider the sequence of open intervals $F_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, for all $n \in \mathbb{N}$. Obviously

(a) $F_1 \supset F_2 \supset F_3 \supset \cdots$, i.e., $\langle F_n \rangle$ is a nest of open intervals

(b) $\delta(F_n) = \left|\frac{1}{n} - \left(-\frac{1}{n}\right)\right| = \frac{2}{n} \to 0$ as $n \to \infty$.

But the intersection $\bigcap_{n=1}^{\infty} F_n = \{0\}$, which contains exactly one point.

(iii) In the real line $\mathbb{R}$, let us consider the sequence of closed intervals $F_n = \left[-1 - \frac{1}{n}, 1 + \frac{1}{n}\right]$, for all $n \in \mathbb{N}$. Obviously

(a) $F_1 \supset F_2 \supset F_3 \supset \cdots$, i.e., $\langle F_n \rangle$ is a nest of closed intervals

(b) $\delta(F_n) = \left|\left(1 + \frac{1}{n}\right) - \left(-1 - \frac{1}{n}\right)\right| = 2 + \frac{2}{n} \to 2(\neq 0)$ as $n \to \infty$.

But the intersection $\bigcap_{n=1}^{\infty} F_n = [-1, 1]$, which contains uncountably infinite number of points.

(iv) In the real line $\mathbb{R}$, let us consider the sequence of sets $F_n = \left[-1 - \frac{1}{n}, -1\right] \cup \left[1, 1 + \frac{1}{n}\right]$, for all $n \in \mathbb{N}$. Obviously

(a) Since union of a finite number of closed sets in closed, it follows that each F_n are closed in the real line $\mathbb{R}$

(b) $\delta(F_n) = \sup\{|\alpha - \beta|; \alpha, \beta \in F_n\} = \left|\left(1 + \frac{1}{n}\right) - \left(-1 - \frac{1}{n}\right)\right| = 2 + \frac{2}{n} \to 2(\neq 0)$ as $n \to \infty$.

But the intersection $\bigcap_{n=1}^{\infty} F_n = \{-1, 1\}$, which is not a singleton set.

(v) In the discrete metric space $(\mathbb{R}, d)$ let us define $F_n = \{x \in \mathbb{R} : x > n\}$; for all $n \in \mathbb{N}$. Obviously,

(a) $\langle F_n \rangle$ is a nested sequence of non-empty close sets in $(\mathbb{R}, d)$

(b) $\delta(F_n) = 1(\neq 0)$ for each n.

But the intersection $\bigcap_{n=1}^{\infty} F_n = \phi$.

(vi) Let $\mathbb{Q}$ be the set of rational numbers, and d the usual metric on $\mathbb{Q}$ and $(\mathbb{Q}, d)$ is not complete. Let $F_n = \left\{x \in \mathbb{Q}^+ : 2 < x^2 \leq 2 + \frac{1}{n}\right\}$; for all $n \in \mathbb{N}$.

(a) $\langle F_n \rangle$ is a sequence of non-empty closed sets in $\mathbb{Q}$

(b) $\delta(F_n) \to 0$.

But the intersection $\bigcap_{n=1}^{\infty} F_n = \phi$.

(vii) Consider the two dimensional Euclidean plane $\mathbb{R}^2$. Let $F_n = S\left((0,0), \frac{1}{n}\right) - \{(0,0)\}$, for all $n \in \mathbb{N}$.

(a) $\langle F_n \rangle$ is a nested sequence of non-empty

(b) $\delta(F_n) \to 0$ as $n \to \infty$.

The point $(0,0)$ is a limit point of F_n, and it is not in F_n. Thus the intersection $\bigcap_{n=1}^{\infty} F_n = \phi$.

THEOREM **15 (Completeness and continuity) Suppose that (X, d_X) and (Y, d_Y) are metric spaces and that (X, d_X) is complete. Suppose there exists a bijective function $f : X \to Y$ such that f is continuous and f^{-1} is uniformly continuous. Then Y is complete.**

Proof: Suppose $\langle a_n \rangle$ is a Cauchy sequence in Y . Then $f^{-1}\langle a_n \rangle$ is Cauchy in X by and so converges in X because X is complete. Therefore $\langle a_n \rangle$, being the image of $f^{-1}\langle a_n \rangle$ under the continuous function f, also converges. Since $\langle a_n \rangle$ is an arbitrary Cauchy sequence in Y , Y is complete.

(i) When the closed interval $[1, \infty]$ is endowed with its usual metric, it is a complete metric space. Now endow it with the inverse metric $(a, b) \mapsto |a^{-1} - b^{-1}|$; this space is not complete because the sequence $\langle n \rangle$ is Cauchy but does not converge. Notice, however, that the identity map from $[1, \infty]$ with the Euclidean metric to $[1, \infty]$ with the inverse metric is continuous and has continuous inverse.

(ii) Let d denote the usual metric on $\mathbb{R}^+$. and m denote the exponential metric $(a, b) \mapsto |e^a - e^b|$. $(\mathbb{R}^+, d)$ is complete, being a closed subspace of $\mathbb{R}$ with its usual metric. The identity map from $(\mathbb{R}^+, d)$ to $(\mathbb{R}^+, m)$ is bijective; it is easily verified that continuity of the exponential function implies continuity of this identity map, and the identity function from $(\mathbb{R}^+, m)$ to $(\mathbb{R}^+, d)$ is a Lipschitz map because $|a - b| \leq |e^a - e^b$ for all $a, b \in \mathbb{R}^+$. So $\mathbb{R}^+$ with the exponential metric is complete.

3.1 Complete Subsets

We will notice that the property of universal closure, unlike that of closure, is independent of any particular enveloping metric space. So whether or not a metric space is being considered as a space in its own right or as a subspace of some larger space is irrelevant when we are talking about completeness.

Definition 7 Suppose (X, d) is a metric space and Y is a subset of X. We say that Y is a *complete subset of* X if, and only if, the metric subspace (Y, d) of (X, d) is a complete metric space. For example,

(i) Because $\mathbb{R}$ is complete, the complete subsets of $\mathbb{R}$ are its closed subsets; that is, precisely those that have the nearest-point property. In particular, every closed interval of $\mathbb{R}$ is complete, and, despite its obvious fragmentation, the Cantor set C is also complete.

(ii) An open subset of a complete metric space is not complete unless it is also closed. However, open subsets can be made into complete spaces by judiciously altering the metric.

THEOREM **16 Let (X, d) be a metric space, and let Y be a subset of X. Then Y, considered as a metric space, is complete if and only if Y is closed.**

Proof: Let Y be a complete subspace of X, we are to show that it is closed in X. Let $y \in Y$ be a limit point of Y, then for any positive integer n, $S(y, 1/n)$ must contain a point y_n of Y. Then $\langle y_n \rangle$ is a Cauchy sequence.

Since Y is complete, the $\langle y_n \rangle$ converges to some element of Y. But $\lim_{n\to\infty} y_n = y$ and so $y \in Y$. Thus we see that Y contains all the accumulation points and consequently Y is closed.

Conversely, let Y be closed and we shall show that Y is complete. Let $\langle y_n \rangle$ be any Cauchy sequence of elements of Y. Since $Y \subseteq X$, $\langle y_n \rangle$ is also a Cauchy sequence in X, and so converge to a point $y_0 \in X$. We now show that $y_0 \in Y$, for this we consider the following two cases :

(i) If the range set of $\langle y_n \rangle$ consists of finite number of distinct points, then $y_n = y_0$ for infinitely many values of n. Since $\langle y_n \rangle$ is in Y, it follows that $y_0 \in Y$.

(ii) If the range set of $\langle y_n \rangle$ has infinitely many distinct points, then y_0 is a limit point of the range set of $\langle y_n \rangle$ and so y_0 is also a limit point of Y. Since Y is closed, $y_0 \in Y$.

Thus we have shown that every Cauchy sequence in Y converges to a point in Y. Hence Y is complete.

Deduction 3.1 Let y be a limit point of Y. Then $\exists$ a sequence $\langle y_n \rangle$ in Y where $y_n \neq y \ \forall \ n$ such that $y_n \to y$ as $n \to \infty$.

$\therefore$ $\langle y_n \rangle$ is a convergent sequence in (X, d).

Since every convergent sequence is Cauchy, $\langle y_n \rangle$ is a Cauchy sequence in (X, d). Also $y_n \in Y$ $\forall n$, $\langle y_n \rangle$ is a Cauchy sequence in (Y, d_y).Thus $y \in Y$. Hence Y is a closed set.

Therefore, if (Y, d_y) be a sub-space of a metric space (X, d) and if Y is complete then Y is a closed sub-set of X.

Theorem **17 (Unions of complete subsets) Let (X, d) is a complete metric space. Prove that if E_1 and E_2 are complete sub-sets of (X, d) then $E_1 \cup E_2$ is also complete.**

Proof: Let $\langle x_n \rangle$ be a Cauchy sequence in $E_1 \cup E_2$. Then at least one of E_1, E_2 contains infinitely many members of $\langle x_n \rangle$. Without loss of generality suppose E_1 contains infinitely many members of $\langle x_n \rangle$. Then $\exists$ a sub-sequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $x_{n_k} \in E_1 \; \forall \; k \in \mathbb{N}$. Since $\langle x_n \rangle$ is Cauchy sequence, $\langle x_{n_k} \rangle$ is a Cauchy sequence in E_1. Since E_1 is complete, $\langle x_{n_k} \rangle$ converges to some point $x \in E_1$.

Thus $\langle x_n \rangle$ is a Cauchy sequence and a sub-sequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ converges to x in $E_1 \cup E_2$. This implies $\langle x_n \rangle$ converges to x in $E_1 \cup E_2$. Hence $E_1 \cup E_2$ is complete.

Note: Union of a finite number of complete subsets of a metric space is complete.

Theorem **18 (Intersections of complete subsets) Intersection of any number of complete sub-sets of a metric space is complete.**

Proof: Let (X, d) be a metric space and $\{F_\alpha : \alpha \in A\}$ be an arbitrary collection of complete sub-sets of X. Suppose $F = \bigcap\limits_{\alpha \in A} F_\alpha$. Since every complete sub-set is closed, for each $\alpha \in A$, F_α is a closed sub-set of X and so F is a closed sub-set of X (as intersection of arbitrary collection of closed set is a closed set).

For a fixed $\alpha_0 \in A, F_\alpha$ is a complete metric space and F is a closed sub-set of F_{α_0}. Since a closed sub-set of a complete metric space in complete, F is complete.

Theorem **19 Let (X, d) be a complete metric space and (Y, d_Y) be a sub-space of (X, d). Then Y is closed implies Y is complete.**

Proof: Let $\langle x_n \rangle$ be a Cauchy sequence in Y. Then $\langle x_n \rangle$ is a Cauchy sequence in (X, d) (since $Y \subseteq X$). But (X, d) being complete, $\langle x_n \rangle$ converges to some point x in X. Let A be the range of the sequence $\langle x_n \rangle$.

Case I: If A is finite then $x_n = x$ for infinitely many values of n. Since $x_n \in Y \; \forall \; n, x \in Y$.

Case II: If A is an infinite set then x is a limit point of A. Also since $A \subseteq Y, s$ a limit point of Y. Since Y is closed, $x \in Y$.

$\therefore$ From the both cases we have $\langle x_n \rangle$ converges to x in Y. Hence (Y, d_Y) is a complete metric space.

Theorem **20 Completeness is preserved under isometries**.

Proof: Let (X, d) and (Y, ρ) be metric spaces. Let $T : X \to Y$ be an isometry from (X, d) onto (Y, ρ) and also let (X, d) be complete. Since every isometry is a one-one onto mapping, hence for each $n \in \mathbb{N}$, $\exists x_n \in X$ such that $f(x_n) = y_n$. Since f is isometry, we obtain

$$d(x_n, x_m) = \rho(T(x_n), T(x_m)) = \rho(y_n, y_m); \forall n, m \in \mathbb{N}. \tag{15}$$

Since $< y_n >$ is a cauchy sequence, hence for giving $\varepsilon > 0$, $\exists \; n_0 \in \mathbb{N}$ such that

$$\rho(y_n, y_m) < \varepsilon \; ; \; m, n \geq n_0$$
$$\Rightarrow \quad d(x_n, x_m) < \varepsilon \; ; \; m, n \geq n_0.$$
$$\Rightarrow \quad \langle x_n \rangle \text{ is a Cauchy Sequence in } X.$$

Since (X, d) is complete and $\langle x_n \rangle$ is a Cauchy Sequence, then $\langle x_n \rangle$ is a convergent sequence. Hence there exists $x \in X$ such that $x_n \to x$. Hence corresponding to $\varepsilon > 0$, $\exists\, n_1 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$, $\forall n \geq n_1$. Since T is an isometry, we have

$$d(x_n, x) = \rho(T(x_n), T(x)) = \rho(y_n, f(x)) < \varepsilon \text{ for } n \geq n_1.$$
$$\Rightarrow \quad y_n \to f(x) : \; n \to \infty.$$

Thus every Cauchy Sequence in y is a convergent sequence and hence (y, ρ) is complete.

Result **7** We know the real line is complete and space $X = (0, 1)$ with usual metric is not complete. Now, the homeomorphic image of $\mathbb{R}$ in X. But $\mathbb{R}$ is complete whereas X is incomplete. Thus homeomorphism need not presence completeness.

4 Completeness and Contraction Mapping

Strong contractions on a metric space, when iterated, tend to pull all the points of the space together into a single point. Banach's Theorem, also called the *Banach Contraction Principle,* is that such a fixed point must exist if the space is complete and can, in that case, be computed by iteration. This theorem is invaluable for developing algorithmic procedures and generally for computing solutions to equations.

Definition 8 Suppose X is a non-empty set and $f : X \to X$. A point $x \in X$ is called a *fixed point* for f if and only if $f(x) = x$. For example,

(i) Let $X = \mathbb{R}$ be a nonempty set and $T : X \to X$ be a mapping defined by $T(x) = x + a$, for some fixed number $a \neq 0$. Then the translation T has no fixed point.

(ii) A rotation of the plane has a single fixed point. Indeed, the centre of rotation is the only fixed point.

(iii) The mapping $T : X \to X$ defined by $T(x) = x/2$. Then $x = 0$ is the only fixed point of T.

(iv) The mapping $T : X \to X$ defined by $T(x) = x^2$. Then $x = 0$ and $x = 1$ are two fixed points of T.

(v) The mapping $T : X \to X$ defined by $T(x) = x$. Then T has infinitely many fixed points. In fact, every point of X is a fixed point.

(vi) The projection $(x, y) \to x$ of $\mathbb{R}^2$ onto the x-axis has infinitely many fixed points. In fact, all points of the x-axis are fixed points.

(vii) Every continuous function from $[0, 1]$ to $[0, 1]$ has at least one fixed point, though it may have many. At least, the following assertion is intuitively true: suppose $f : [0, 1] \to [0, 1]$ is continuous, $f(0) = \alpha \in (0, 1]$ and $f(1) = \beta \in [0, 1)$. Then the graph of f joins $(0, \alpha)$ continuously to $(1, \beta)$ and must cross the line $\{x \in \mathbb{R}^2 : x_1 = x_2\}$ somewhere along the way (Fig. 1).

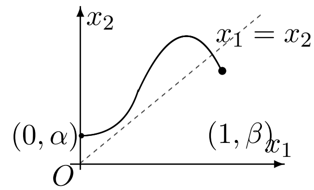

Figure 1: The graph $x_1 = x_2$.

The examples show that a mapping may not have any fixed point, it may have unique fixed point, it may have more than one or even infinitely many fixed points.

Definition 9 Let (X, d) be a complete metric space and $T : X \to X$ be a mapping (or called an *operator*). Then T is said to be

(i) *non-expansive*, if

$$d(T(x), T(y)) \leq d(x, y), \quad \forall x, y \in X. \tag{16}$$

(ii) *contractive* or *weak contraction mapping* or *contractible* or *contractive* or *shrinking map* if

$$d(T(x), T(y)) \leq d(x, y), \quad \forall x, y \in X, x \neq y. \tag{17}$$

(iii) a *contraction mapping* or *contraction on* X if there exists a real number α with $0 \leq \alpha < 1$ such that

$$d\Big(T(x), T(y)\Big) \leq \alpha\ d(x, y) < d(x, y); \qquad \forall x, y \in X, x \neq y. \tag{18}$$

The number α is usually referred as *Lipschitz constant* of T.

It is stressed that α in this definition (18) is independent of x, y in X. Thus in a contraction mapping, the distance between the images of any two points is less than the distance between the points. Hence the application of T to each of two points 'contracts' the distance between them. Every contraction mapping is uniformly continuous.

Theorem **21 (Banach fixed point theorem) Every contraction mapping on a complete metric space has a unique fixed point.**

Proof: Existence: Let (X, d) be a complete metric space and x, y be any two points in X. Let $T : X \to X$ is a contraction on X, there exists a $\alpha \in \mathbb{R}$ with $0 \leq \alpha < 1$ such that

$$\begin{aligned} & d\Big(T(x), T(y)\Big) \leq \alpha\ d(x, y); \qquad \forall x, y \in X \\ \therefore \quad & d\Big(T^2(x), T^2(y)\Big) \leq d\Big(T(x), T(y)\Big) \leq \alpha^2\ d(x, y);\ \forall x, y \in X \\ & \qquad \vdots \qquad\qquad\qquad \vdots \\ & d\Big(T^n(x), T^n(y)\Big) \leq d\Big(T^{n-1}(x), T^{n-1}(y)\Big) \leq \alpha^n\ d(x, y); \forall x, y \in X. \end{aligned} \tag{19}$$

Now let x_0 be any point of X and inductively construct the sequence $\langle x_n \rangle$ of points in X as:

$$x_1 = T(x_0), x_2 = T(x_1) = T^2(x_0), \cdots, x_n T(x_{n-1}) = \cdots = T^n(x_0), \cdots.$$

Clearly, $\langle x_n \rangle$ is the sequence of images of x_0 under repeated application of T. We claim that $\langle x_n \rangle$ is a Cauchy sequence. For, let m, n be any positive integers such that $m > n$. Write $m = n + p$, where p is any positive integers greater than or equal to 1. Using triangle inequality, we have

$$\begin{aligned} d(x_n, x_m) &= d(x_n, x_{n+p}) \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\le d\Big(T^n(x_0), T^n(x_1)\Big) + d\Big(T^{n+1}(x_0), T^{n+1}(x_1)\Big) \\ &\qquad + \cdots + d\Big(T^{n+p-1}(x_0), T^{n+p-1}(x_1)\Big); \quad \text{Eq. (19)} \\ &\le \alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1) + \cdots + \alpha^{n+p-1} d(x_0, x_1) \\ &\le \alpha^n d(x_0, x_1)\Big[1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1}\Big] = \frac{\alpha^n}{1-\alpha} d(x_0, x_1). \end{aligned} \tag{20}$$

Since $0 \le \alpha < 1$, so $\lim_{n\to\infty} \alpha^n = 0$, it follows that $d(x_n, x_m)$ can be made less than any pre-assigned positive number ε by taking n (and hence m) sufficiently large. Thus $\langle x_n \rangle$ is a Cauchy sequence. But by hypothesis X is complete, so that there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty} T(x_n) = T(x). \tag{21}$$

Now $T(x_n) = x_{n+1}$ so that $\langle T(x_n) \rangle$ is a subsequence of $\langle x_n \rangle$ and consequently $\langle T(x_n) \rangle$ must also converge to x, i.e., $\lim_{n\to\infty} T(x_n) = x$. Using Eq. (21), we have

$$\lim_{n\to\infty} T(x_n) = T(x) = x. \tag{22}$$

This shows that x is a fixed point.
Uniqueness : All that remain to show is that if $y \in X, y \ne x$, then y cannot be a fixed point. Suppose, if possible, y is a fixed point. Then $T(y) = y$. Since, $T(x) = x$, we have

$$\begin{aligned} & d(x, y) = d(T(x), T(y)) \le \alpha\, d(x, y) \\ \Rightarrow \quad & \alpha\, d(x, y) \ge d(x, y) > 0; \quad \text{as } d(x, y) \ne 0 \\ \Rightarrow \quad & \alpha \ge 1 \end{aligned}$$

which is a contradiction. Hence $T(y) \ne y$ and the proof is complete.

(i) This theorem is often stated as ‘ T has precisely one fixed point’.

(ii) Let $T : (X, d) \to$ itself be a contraction, and $\varepsilon > 0$ be given. Let us take a $\delta > 0$ to satisfy $\delta < \varepsilon/2\alpha$. Now, if $d(x, y) < \delta$, we have

$$d(T(x), T(y)) \le \alpha\, d(x, y) < \alpha\delta < \varepsilon/2 < \varepsilon.$$

Hence T is uniformly continuous over (X, d).

(iii) If the condition of completeness in Banach fixed point theorem be dropped, then T may not have a fixed point. For example, let $X = (0, 1)$ and the mapping $T : X \to X$ defined by $T(x) = x/2$, then $T(0) = 0 \notin X$. Here X is not a complete metric space with the usual metric and T does not have any fixed point.

(iv) Let $X = \{x \in \mathbb{R} : x \geq 1\}$ be a complete metric space with usual metric of reals and the function $T : X \to X$ be defined as, $T(x) = \dfrac{x}{2} + \dfrac{1}{x}$ for $x \in X$. Let $x, y \in X$, then

$$\begin{aligned} |T(x) - T(y)| &= \left|\frac{x}{2} + \frac{1}{x} - \frac{y}{2} + \frac{1}{y}\right| = \left|\frac{1}{2}(x-y) - \left(\frac{1}{y} - \frac{1}{x}\right)\right| \\ &= \left|\frac{1}{2}(x-y) - \frac{x-y}{xy}\right| = \left|(x-y)\left(\frac{1}{2} - \frac{1}{xy}\right)\right| \\ &= |x-y| \left|\frac{1}{2} - \frac{1}{xy}\right|. \end{aligned}$$

Since $x \geq 1$ and $y \geq 1$, we have $1/xy \leq 1$. So $1/xy$ lies either between 0 and 1/2 or between 1/2 and 1 and in each of these cases

$$\left|\frac{1}{2} - \frac{1}{xy}\right| \leq \frac{1}{2} \Rightarrow |T(x) - T(y)| \leq \frac{1}{2}\,|x-y|.$$

So T is a contraction and Banach contraction principle says that T has unique fixed point x^* given by $x^* = \frac{x^*}{2} + \frac{1}{x^*}$ or $x^* = 2$ or $x^* = \sqrt{2}$.

(v) If T is not contraction in Banach fixed point theorem, then it may not have a fixed point. For example, consider the metric space $X = [1, \infty)$ with the usual metric and the mapping $T : X \to X$ given by $T(x) = x + \frac{1}{x}$. Now, X is a complete metric space but T is not a contraction mapping. In fact

$$|T(x) - T(y)| = |x-y|\left[1 - \frac{1}{xy}\right] < |x-y|;\ \ \forall x, y \in X$$

and so T is contractive. Of course, T does not have any fixed point.

(vi) Consider the complete metric space $X = [0, \infty)$ equipped with the metric of absolute value and consider the mapping $T : X \to X$ given by $T(x) = 1/(1 + x^2)$. Then

(a) the mapping T satisfies $d(T(x), T(y)) < d(x, y)$ and hence T is a contractive map, while T is not a contraction.

(b) T has no fixed point.

(vii) The mapping $T : (0, \frac{1}{3}] \to (0, \frac{1}{3}]$ defined by $T(x) = x^2$ is a contraction, but it has no fixed point in $(0, \frac{1}{3}]$.

(viii) Let $X = \mathbb{R}$ be the real number space and for $x \in \mathbb{R}$, let $T : X \to X$ be defined as $T(x) = x + \frac{\pi}{2} - \tan^{-1} x$, then T is not a contraction mapping.

THEOREM **22** **Let (X, d) be a compact metric space and $T : X \to X$ a contractive map (not necessarily contraction). Then, T has a unique fixed point in X.**

Proof: Existance : Define a mapping $f : X \to \mathbb{R}$ by $f(x) = d(x, T(x))$. We first show that f is continuous. Let $\varepsilon > 0$ be chosen arbitrary. Then

$$\begin{aligned} |f(x) - f(y)| &= \Big|d(x, T(x)) - d(y, T(y))\Big| \\ &\leq \Big|d(x, T(x)) - d(T(x), y)\Big| + \Big|d(T(x), y) - d(y, T(y))\Big| \\ &\quad \text{(by triangle inequality)} \\ &\leq d(x, y) + d(T(x), T(y)) \leq 2d(x, y), \because T \text{ is contractive} \\ &< \varepsilon,\ \text{provided } d(x, y) < \delta < \varepsilon/2. \end{aligned}$$

Since f is continuous and X is compact, $\exists x \in X$ such that $f(x) \leq f(y)$, $\forall y \in X$, i.e., f attains minimum at x. We, now, show that x is a fixed point of T.

Let, if possible, x be not a fixed point of T. Then $Tx \neq x$. Since $f(x) \leq f(y)$, $\forall y \in X$, taking $y = T(x)$, we have

$$\begin{aligned} & f(x) \leq f(T(x)) \\ \Rightarrow \quad & d(x, T(x)) \leq d(T(x), T(T(x))) < d(x, T(x)), \because T \text{ is contractive and } x \neq T(x). \end{aligned}$$

which is impossible. Hence x is a fixed point of T.
Uniqueness : Let, if possible, x and y be two fixed points of T in X. Then $T(x) = x$ and $T(y) = y$. Now,

$$\begin{aligned} & d(x, y) = d(T(x), T(y)) \leq \alpha\ d(x, y) \\ \Rightarrow \quad & d(x, y) = 0 \Rightarrow x = y. \end{aligned}$$

This completes the proof of this theorem.

THEOREM **23 Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If T^m is a contraction on X for some positive integer m, then T has a unique fixed point.**

Proof: Let $S = T^m$. Under the hypothesis, S is a contraction. By Banach Contraction Theorem, the mapping S has unique fixed point, say x^*. Thus

$$S(x^*) = x^* \Rightarrow S^n(x^*) = x^*.$$

Again, in view of Banach Contraction Theorem, we have

$$\lim_{n \to \infty} S^n(x^*) = x^*, \quad \forall x \in X.$$

In particular, taking $x = T(x)$, we get

$$\begin{aligned} x^* &= \lim_{n \to \infty} S^n(x^*) = \lim_{n \to \infty} S^n(T(x^*)) \\ &= \lim_{n \to \infty} T(S^n(x^*)) = \lim_{n \to \infty} T(x^*) = T(x^*). \end{aligned}$$

This shows that x^* is a fixed point of T. Also, since every fixed point of T is a fixed point of $S = T^m$, it follows that T cannot have more than one fixed point and uniqueness follows.

Banach fixed point theorem has many applications in analysis.

(i) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and there exists $k \in [0, 1)$ such that $|f'(x)| \leq \alpha$ for all $x \in \mathbb{R}$. Then, for each $x, y \in \mathbb{R}$ with $x < y$, it follows from the Mean Value Theorem that there exists $c \in (x, y)$ with $f(y) - f(x) = (y - x)f'(c)$. From this we get $|f(y) - f(x)| \leq \alpha|y - x|$, so that f is a strong contraction. Then Banach's Theorem tells us that f has a unique fixed point.

(ii) The graph of the cosine function clearly crosses the line $y = x$ somewhere between $x = 0$ and $x = \pi/2$, (Fig. 2) and it does so exactly once. In other words, the cosine function restricted to $[0, \pi/2]$ has a unique fixed point. It turns out that we can apply Banach's Theorem to discover it despite the fact that the cosine function is not a strong contraction on $[0, \pi/2]$.

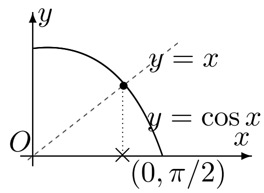

Figure 2: The graph $y = x$ and $y = \cos x$.

(iii) Let $[a, b]$ be a closed interval, which is a complete metric space with usual metric of reals and $g : [a, b] \to [a, b]$ be a function with $g'(x)$ satisfying $|g'(x)| \le \alpha < 1$ in $a \le x \le b$. If $x_1, x_2 \in [a, b]$ with $x_1 < x_2$ we have by Mean-value theorem of differential calculus

$$\begin{aligned} & g(x_2) - g(x_1) = (x_2 - x_1)\ g'(\xi);\ \ x_1 < \xi < x_2 \\ \Rightarrow \quad & |g(x_2) - g(x_1)| = |(x_2 - x_1)|\ |g'(\xi)| \le \alpha |x_2 - x_1|;\ \ 0 < \alpha < 1. \end{aligned}$$

So g is a contraction and Banach Contraction Principle says that there is exactly one member $x^* \in [a, b]$ satisfying $g(x^*) = x^*$ or $g(x^*) - x^* = 0$ or $g(x) - x = 0$ has exactly one root in $[a, b]$.

(iv) The Cantor set is an example of a fractal. On the Fig. 3 of fractal, *Sierpi Lnski's triangle,*

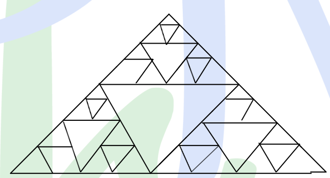

Figure 3: Sierpi Lnski's triangle.

this time in $\mathbb{R}^2$ rather than $\mathbb{R}$. The picture was obtained by iterating the function

$$A \to \Big\{(.5a_1 + c_1, .5a_2 + c_2) : a \in A, c \in \{(1, 1), (1, 50), (50, 50)\}\Big\}$$

on the space of non-empty closed bounded subsets of $\mathbb{R}^2$ starting at a single point.

4.1 Applications of Banach's Fixed Point Theorem

Banach's fixed point theorem has wide and important applications in diversified fields of study.

(i) The most interesting applications of fixed point theorem arise when the underlying metric space is a function space. Here we discuss the existence and uniqueness of the Volterra and Fredehelom integral equations.

(ii) We give an application of Banach fixed point theorem to prove a theorem of Picard on the existence and uniqueness of solutions of a certain class of first order ordinary differential equations.

Theorem **24** **Let ϕ be a function in $\mathcal{C}[a, b]$, and K be a continuous real-valued mapping defined on the square $[a, b] \times [a, b]$ with subspace metric of $\mathbb{R}^2$. Then there exists a real number λ such that the** *Volterra equation*

$$f(x) = \phi(x) + \lambda \int_a^x K(x, y)\ f(y)\ dy;\ \textbf{for all } x \in [a, b]. \tag{23}$$

has a unique solution in $\mathcal{C}[a, b]$.

Proof: Let K be a continuous function on $[a, b] \times [a, b]$ and let ϕ be a continuous function on $[a, b]$. For each parameter $\lambda \in \mathbb{R}$, consider the *Volterra equation* (23). Let $X = \mathcal{C}[a, b]$ be the set of all continuous real-valued functions defined on $[a, b]$ with the uniform metric. Since K is continuous, $\exists$ a constant $M > 0$ such that

$$|K(x, y)| \leq M; \text{ for all } x, y \in [a, b].$$

Define the transformation $T : f \to T(f)$ on X by

$$T(f(x)) = \phi(x) + \lambda \int_a^x K(x, y)\ f(y)\ dy. \tag{24}$$

For all $f, g \in X$, we have

$$\begin{aligned} |T(f(x)) - T(g(x))| &= \left|\lambda \int_a^x K(x, y)\ |f(y) - g(y)|\ dy\right| \\ &\leq |\lambda|\ M(x - a)\ d(f, g); \ \ \forall x \in [a, b]. \end{aligned}$$

Since $T^2(f) - T^2(g) = T(T(f) - T(g))$, we have

$$\begin{aligned} |T^2(f(x)) - T^2(g(x))| &= \left|\lambda \int_a^x K(x, y)\ |T(f(y)) - T(g(y))|\ dy\right| \\ &\leq |\lambda| \int_a^x |K(x, y)|\ |\lambda|\ M(y - a) d(f, g)\ dy \\ &\leq |\lambda|^2 k^2 \int_a^x (y - a) dy\ d(f, g) \\ &\leq \frac{|\lambda|^2 M^2 (x - a)^2}{2} d(f, g). \end{aligned}$$

Continuing this iterative process, we obtain

$$|T^n(f(x)) - T^n(g(x))| \leq \frac{|\lambda|^n M^n (x - a)^n}{n!} d(f, g); \ \ \forall x \in [a, b].$$

$$\therefore \quad |T^n(f) - T^n(g)| \leq \frac{[|\lambda| M(b - a)]^n}{n!}\ d(f, g).$$

Recalling that $\frac{r^n}{n!} \to 0$ as $n \to \infty$ for any $r \in \mathbb{R}$, we conclude that there exists n such that T^n is a contraction mapping. Taking n sufficiently large to have $\frac{[|\lambda| M(b-a)]^n}{n!} < 1$. Hence there exists an unique solution $f \in X$ satisfying $T(f) = f$. Obviously, if $T(f) = f$, then f solves Eq.(23).

Theorem **25 (Picard's theorem) Let $\mathcal{D}$ be a nonempty open subset of the Euclidean plane $\mathbb{R}^2$, $f : \mathcal{D} \to \mathbb{R}$, $\frac{\partial f}{\partial y}$ be continuous maps which satisfy the Lipschitz condition in the second variable**

$$|f(x, y_1) - f(x, y_2)| \leq \alpha\ |y_1 - y_2|, \tag{25}$$

for all $(x, y_1), (x, y_2) \in \mathcal{D}$ and for some $\alpha > 0$, and let $(x_0, y_0) \in \mathcal{D}$. Then the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{26}$$

has an unique solution $y = g(x)$ which passes through the point (x_0, y_0).

Proof: Since $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous in $\mathcal{D}$ and $\mathcal{D}$ is a compact subset of $\mathbb{R}^2$; $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are bounded, and so there exists real numbers M and N such that

$$|f(x,y)| \leq M \text{ and } \left|\frac{\partial f}{\partial y}\right| \leq N; \ \ \forall (x,y) \in \mathcal{D}. \tag{27}$$

Now, if (x, y_1) and (x, y_2) be any points in $\mathcal{D}$, then by Lagrange's mean value theorem of calculus

$$\begin{aligned} & |f(x,y_1) - f(x,y_2)| = |y_1 - y_2| \left|\frac{\partial f}{\partial y}(x, y_1 + (y_1 - y_2)\theta)\right|; \ \ 0 < \theta < 1. \\ \Rightarrow \quad & |f(x,y_1) - f(x,y_2)| \leq |y_1 - y_2| \cdot N; (x,y_1),(x,y_2) \in \mathcal{D}. \end{aligned} \tag{28}$$

Now, if $y = g(x)$ is the solution of the differential equation (26), then

$$g'(t) = f(t, g(t)); \quad \text{and } g(x_0) = y_0. \tag{29}$$

Then integrating Eq.(29) yields

$$\begin{aligned} & g(x) - g(x_0) = \int_x^{x_0} f(t, g(t)) \ dt \\ \Rightarrow \quad & g(x) = g(x_0) + \int_x^{x_0} f(t, g(t)) \ dt. \end{aligned} \tag{30}$$

Conversely, if $y = g(x)$ satisfies Eq. (30), then it follows that $g(x_0) = y_0$. Thus $y = g(x)$ such that $y_0 = g(x_0)$ is a solution of Eq. (26) if and only if it satisfies the integral equation, Eq. (30). Therefore, it is sufficient to prove that Eq. (30) has a unique solution. Choose a positive constant δ such that $N\delta < 1$ and the closed rectangle

$$\mathcal{D}' = \Big\{(x,y) \in \mathbb{R}^2 : |x - x_0| \leq \delta \text{ and } |y - y_0| \leq M\delta\Big\} \subset \mathcal{D}.$$

Let X be the set of all real-valued functions $y = g(x)$ which is defined and continuous in $|x - x_0| \leq \delta$, such that

$$d(g(x), y_0) = |g(x) - y_0| \leq M\delta.$$

Let set X is a closed subspace of the complete metric space of continuous functions $C[x_0 - \delta, x_0 + \delta]$ with the sup metric d. Let ψ in $C[x_0 - \delta, x_0 + \delta]$ be a limit point of X, then $\exists$ a sequence $\langle g_n \rangle$ in X which converges in $C[x_0 - \delta, x_0 + \delta]$ to ϕ. So corresponding to a real number $\varepsilon > 0, \exists k \in \mathbb{N}$ such that

$$d(g_n, \psi) = |g_n(x) - \psi(x)| < \varepsilon; \ \ \forall n \geq k \text{ and } x \in [x_0 - \delta, x_0 + \delta].$$

Using Triangle inequality, we get

$$|\psi(x) - y_0| \leq |\phi(x) - g_n(x)| + |g_n(x) - y_0| < \varepsilon + M\delta.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$|\psi(x) - y_0| \leq M\delta; \ \ \forall x \in [x_0 - \delta, x_0 + \delta].$$

Thus $\psi \in X$ and ψ is a closed set. Thus X is complete. Consider a mapping $T : X \to X$ defined by

$$T(g) = h; \quad \text{where } h(x) = y_0 + \int_x^{x_0} f(t, g(t))\, dt.$$

Now $h(x)$ is a continuous function on $[x_0 - \delta, x_0 + \delta]$ and

$$d(h(x), y_0) = \sup \Big| \int_x^{x_0} f(t, g(t))\, dt \Big| \leq M(x - x_0) \leq M \cdot \delta,$$

shows that $h(x) \in X$ and T is well-defined. Now

$$\begin{aligned} d(T(g), T(g_1)) &= d(h, h_1) = \sup \Big| \int_x^{x_0} \Big[f(t, g(t)) - f(t, g_1(t)) \Big]\, dt \Big| \\ &\leq \int_x^{x_0} \sup \Big| f(t, g(t)) - f(t, g_1(t)) \Big|\, dt \\ &\leq \alpha \int_x^{x_0} |g(t) - g_1(t)|\, dt \\ &\leq \alpha \cdot d(g, g_1)(x - x_0) \leq \alpha\delta d(g, g_1) \\ \therefore d(T(g), T(g_1)) &\leq \beta d(g, g_1); \quad 0 \leq \beta = \alpha\delta < . \end{aligned}$$

Thus is contraction mapping on X into itself. Thus by Banach contraction theorem, the contraction mapping T has a unique fixed point $g \in X$, i.e., $T(g) = g$. This means that there is a unique solution $g \in X$ as given in Eq. (30).

Theorem 26 **(Implicit Function) Let $\mathcal{D}$ be an open set in $\mathbb{R}^2$ containing a point (x_0, y_0) and $f : \mathcal{D} \to \mathbb{R}$ be a mapping such that**

(i) $f(x, y)$ **is continuous in** $\mathcal{D}$**,**

(ii) $\dfrac{\partial f}{\partial x}$ **and** $\dfrac{\partial f}{\partial y}$ **are both continuous in** $\mathcal{D}$**, and**

(iii) $f(x_0, y_0) = 0$ **and** $\dfrac{\partial f}{\partial y}\Big|_{(x_0, y_0)} \neq 0$**.**

Then $\exists$ **a rectangle** $\mathcal{R} = \Big\{(x, y) : |x - x_0| < h, |y - y_0| < k\Big\} \subset \mathcal{D}$ **and a function** $g : [x_0 - h, x_0 + h] \to [y_0 - k, y_0 + k]$ **such that**

(i) $f(x, g(x)) = 0; \quad \forall x \in [x_0 - h, x_0 + h]$, **and**

(ii) g **is differentiable in** $[x_0 - h, x_0 + h]$ **and** $g'(x) = -\Big(\dfrac{\partial f}{\partial x}\Big) \Big/ \Big(\dfrac{\partial f}{\partial y}\Big)\Big|_{(x, g(x))}$**.**

Proof: Let $\dfrac{\partial f}{\partial x}\Big)_{(x_0, y_0)} = p$ and $\dfrac{\partial f}{\partial y}\Big)_{(x_0, y_0)} = q$. Define a mapping $F : \mathcal{D} \to \mathbb{R}$ by

$$F(x, y) = y - \frac{1}{q} f(x, y); \quad \forall (x, y) \in \mathcal{D}.$$

Then $F(x,y)$ is continuous in $\mathcal{D}$; $F(x_0,y_0) = y_0$ and $\left.\frac{\partial F}{\partial y}\right)_{(x_0,y_0)} = 0$. Since $\frac{\partial F}{\partial y}$ is continuous in $\mathcal{D}$, $\exists h,k \in \mathbb{R}^+$ such that

$$\frac{\partial F}{\partial y} < \frac{1}{2};\ \ \forall (x,y) \in \mathcal{R}.$$

Since $F(x,y)$ is continuous and $F(x_0,y_0) = y_0$, we can take smaller h if necessary such that

$$F(x,y_0) - y_0| < \frac{1}{2}k;\ \ \forall x \in [x_0-h, x_0+h].$$

Now, let X be the set of all continuous real valued functions

$$g : [x_0-h, x_0+h] \to [y_0-k, y_0+k];\ \text{with } g(x_0) = y_0.$$

X is obviously non-empty. Let $d : X \times X \to \mathbb{R}$ be a mapping defined by

$$d(g_1,g_2) = \sup\Big\{|g_1(x) - g_2(x)| : \forall x \in [x_0-h, x_0+h]\Big\},$$

then (X,d) is a complete metric space. Now, we define a mapping $T : X \to X$ by

$$T(g) = \psi,\ \text{where } \psi(x) = f(x,g(x));\ \ \forall x \in [x_0-h, x_0+h]\Big\}.$$

Then ψ is continuous in $[x_0-h, x_0+h]$, $\phi(x_0) = f(x_0, g(x_0)) = y_0$ and using Lagrange mean-value theorem

$$\begin{aligned} |\psi(x) - y_0| &= |f(x,g(x)) - y_0| \\ &\le |f(x,g(x)) - f(x,y_0)| + |f(x,y_0) - y_0| \\ &\le \Big|\frac{\partial f}{\partial y}\Big|_{(x,\xi)} |g(x) - y_0| + \frac{1}{2}k; \end{aligned}$$

for some ξ between $g(x)$ and y_0. Therefore $|h(x) - y_0| \le k$. Thus T is induced a mapping from X to itself. Also,

$$\begin{aligned} |\psi_1(x) - \psi_2(x)| &= \Big|f(x,g_1(x)) - f(x,g_2(x))\Big| \\ &\le \Big|\frac{\partial f}{\partial y}\Big|_{(x,\eta)} |g_1(x) - g_2(x)| \le \frac{1}{2}|g_1(x) - g_2(x)|. \end{aligned}$$

This shows that T is a contraction marring. Therefore, T has a unique fixed point g in X so that $T(g) = g$, that is, $g(x) = F(x,g(x))$. Thus $f(x,g(x)) = 0;\ \ \forall x \in [x_0-h, x_0+h]$.

Now it remains to prove that g is differentiable. Let x and $x + \triangle x$ be any two points in $[x_0-h, x_0+h]$. Then $f(x,g(x)) = 0 = f(x+\triangle x, g(x+\triangle x))$, and

$$\begin{aligned} 0 &= f(x+\triangle x, g(x+\triangle x)) - f(x,g(x)) \\ &= f(x+\triangle x, g(x+\triangle x)) - f(x+\triangle x, g(x)) + f(x+\triangle x, g(x)) - f(x,g(x)) \\ &= \frac{\partial f}{\partial y}\Big|_{(x+\triangle x,\xi')} [g(x+\triangle x) - g(x)] + \frac{\partial f}{\partial x}\Big|_{(\eta',g(x))} \triangle x \end{aligned}$$

for some ξ' between $g(x+\triangle x)$ and $g(x)$, and η' between x and $x+\triangle x$. Therefore,

$$\frac{g(x+\triangle x) - g(x)}{\triangle x} = -\Big(\frac{\partial f}{\partial x}\Big)_{(\eta',g(x))} \Big/ \Big(\frac{\partial f}{\partial y}\Big)_{(x+\triangle x,\xi')}$$

$$\therefore\quad g'(x) = \lim_{\triangle x \to 0} \frac{g(x+\triangle x) - g(x)}{\triangle x} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}\Big)_{(x,g(x))}.$$

Example 18 **Let $X = \mathbb{R}^n$ being the set of n-rowed column vectors of real numbers, is a metric space with the metric $d_\infty(x, y) = \max_{1\le i\le n} |x_i - y_i|$. Find the condition for the system of equations**

$$y = Ax + b;\ \textbf{where, } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}. \tag{31}$$

and $x = (x_1\ x_2 \cdots x_n)^T$; $b = (b_1\ b_2 \cdots b_n)^T$ to have precisely one solution.

Proof: Let us consider the space $\mathbb{R}^n$, with the metric is given by $d_\infty(x, y) = \max_{1\le i\le n} |x_i - y_i|$. On $\mathbb{R}^n$, let us define $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $y = T(x) = Ax + b$. Let $u = (u_1\ u_2 \cdots u_n)^T$ and $v = (v_1\ v_2 \cdots v_n)^T$ be any two elements of X. Then

$$\begin{aligned} d_\infty(T(u), T(v)) &= \max_{1\le i\le n} \Big|a_{ij}(u_i - v_j)\Big| \\ &\le \max_{1\le i\le n} |u_i - v_j| \ \max_{1\le i\le n} \Big|a_{ij}\Big| \\ &\le d_\infty(u, v)\ \ \alpha;\ \ \alpha = \max_{1\le i\le n} \Big|a_{ij}\Big|. \end{aligned}$$

Assume that $\alpha = \max_{1\le i\le n} \Big|a_{ij}\Big| < 1$, then T is a contraction mapping, and so Banach Contraction Theorem, there is a unique fixed point $x \in X$ of T. Then the system of linear equations $x = Ax + b$ and the system of linear equations (31) has an unique solution x.

Consider the following system of n linear equations with n unknowns:

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \quad \cdots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \tag{32}$$

Taking, $x = (x_1\ x_2 \cdots x_n)^T$ and $b = (b_1\ b_2 \cdots b_n)^T$, then in matrix notation, Eq.(32) can be written as $Ax = b$, where A is given in Eq. (31). This system (32) can be written as

$$\left.\begin{array}{c} x_1 = (1 - a_{11})x_1 - a_{12}x_2 - \cdots - a_{1n}x_n + b_1 \\ x_2 = -a_{21}x_1 + (1 - a_{22})x_2 - \cdots - a_{2n}x_n + b_2 \\ \vdots \quad \cdots \quad \vdots \quad \vdots \\ x_n = -a_{n1}x_1 - a_{n2}x_2 - \cdots (1 - a_{nn})x_n + b_n \end{array}\right\} \tag{33}$$

5 Incomplete Metric Space

A metric space (X, d) is called *incomplete* if it is not complete. If (X, d) is an incomplete metric space, then we say that d is an incomplete metric on X. To justify the motivation and meaningfulness of the above definition, we must cite examples of incomplete metric spaces:

(i) Let the $\mathcal{P}[a,b]$ be the set of all polynomials defined on $[a,b]$ (a subspace of the metric space $C[a,b]$) with the uniform metric d_∞ defined in it as

$$d_\infty(f,g) = \max_{t\in[a,b]} |f(t)-g(t)|; \quad f,g \in \mathcal{P}[a,b].$$

Let us take $a=0$ and $b=1$. Consider the following sequence

$$f_n(t) = \sum_{k=0}^{n} \left(\frac{t}{2}\right)^k = 1 + \frac{t}{2} + \frac{t^2}{2^2} + \cdots + \frac{t^n}{2^n}; \quad \forall t \in [0,1].$$

Clearly, $f_n(t) \in \mathcal{P}[0,1]$ for each $n \in \mathbb{N}$. Taking $m<n$ and observe that

$$\begin{aligned} d_\infty(f_n,f_m) &= \max_{t\in[0,1]} |f_n(t)-f_m(t)| \\ &= \max_{t\in[0,1]} \Big| \sum_{k=0}^{n} \left(\frac{t}{2}\right)^k - \sum_{k=0}^{m} \left(\frac{t}{2}\right)^k \Big| = \max_{t\in[0,1]} \Big| \sum_{k=m+1}^{n} \left(\frac{t}{2}\right)^k \Big| \\ &\le \max_{t\in[0,1]} \Big| \sum_{k=m+1}^{n} \left(\frac{1}{2}\right)^k \Big| = \frac{1}{2^m} - \frac{1}{2^n}. \end{aligned}$$

This difference is arbitrarily small for large enough m and n, which implies that $\langle f_n \rangle$ is a Cauchy sequence in $\mathcal{P}[0,1]$. However, this sequence does not converge in $(\mathcal{P}[0,1], d_\infty)$, because

$$\lim_{n\to\infty} f_n(t) = \frac{2}{2-t}, \text{ for all } t \in [0,1]$$

is not a polynomial. Thus $(\mathcal{P}[0,1], d_\infty)$ is not complete. Also, in view of the Weierstrass polynomial approximation theorem, the uniform limit of a sequence of polynomials need not be a polynomial and so $(\mathcal{P}[a,b], d_\infty)$ is not a complete metric space.

(ii) Consider the sequence $\langle x_n \rangle$ of rational numbers in the usual metric space $\mathbb{Q}_u$ as follows

$$x_1 = 1.7, x_2 = 1.73, x_3 = 1.732, x_4 = 1.73205\cdots$$

The sequence $\langle x_n \rangle$ converges to $\sqrt{3}$. Hence the sequence $\langle x_n \rangle$ is a Cauchy sequence. However, it does not converge to any point of $\mathbb{Q}_u$ so $\mathbb{Q}$ is not complete.

(iii) Consider the space $X = C[0,1]$, with the metric

$$d_1(x,y) = \int_0^1 |x(t)-y(t)|\, dt; \quad x,y \in [0,1]$$

then we know (X, d_1) is a metric space. Consider $\langle x_n \rangle$ in X, where

$$x_n(t) = \begin{cases} 0; & 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ nt - \frac{1}{2}n + 1; & \frac{1}{2} - \frac{1}{n} < t \le \frac{1}{2} \\ 1; & \frac{1}{2} < t \le 1 \end{cases}$$

which can be represented graphically as in Fig. 4. Now we are to show that $\langle x_n \rangle$ is a

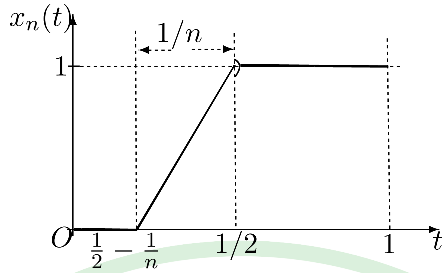

Figure 4: Graph of $x_n(t)$

Cauchys sequence.

$$\begin{aligned} d_1(x_m, x_n) &= \int_0^1 |x_m(t) - x_n(t)|\ dt \\ &\leq \int_{1/2-1/m} x_m(t)\ dt + \int_{1/2-1/n}^{1/2} x_n(t)\ dt = \frac{1}{2}\Big(\frac{1}{m} + \frac{1}{n}\Big) \\ &< \varepsilon; \forall m, n > n_0,\ \text{where } n_0 = [2/\varepsilon] + 1 \in \mathbb{N}. \end{aligned}$$

This shows that $\langle x_n \rangle$ is a Cauchy sequence. Geometrically, the measure $d_1(x_m, x_n)$ repre-

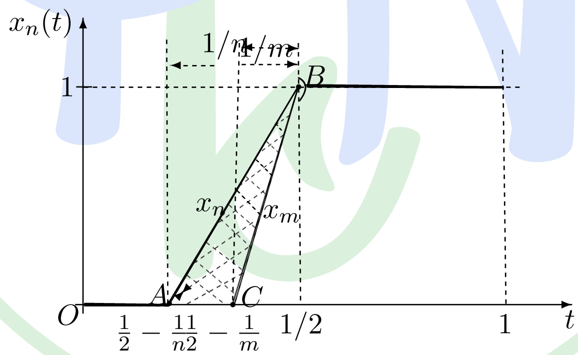

Figure 5:

sents the area of the triangle as shown in the Fig. 5. Suppose that there is a function $x(t)$ such that $d_1(x_n, x) \to 0$. But

$$d_1(x_n, x) = \int_0^{1/2-1/n} |x(t)|dt + \int_{1/2-1/n}^{1/2} |x_n(t) - x(t)|dt + \int_{1/2}^1 |1 - x(t)|dt. \tag{34}$$

Since the integrands are non-negative, so is the each integral on the right hand side of Eq. (34). Since $d_1(x_n, x) \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \int_0^{1/2-1/n} |x(t)|dt = 0 \text{ and } \int_{1/2}^1 |1 - x(t)|dt = 0. \tag{35}$$

Since f is continuous, we have

$$x(t) = \begin{cases} 0; & \text{if } 0 \leq t < 1/2 \\ 1; & \text{if } 1/2 \leq t \leq 1 \end{cases} \tag{36}$$

Therefore, x is not continuous which contradicts to our supposition that x is a continuous function. Hence $C[0, 1]$ is not complete.

(iv) Consider the space of all natural numbers $\mathbb{N}$ with the metric $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$, for all $x, y \in \mathbb{N}$. Let $\langle n \rangle_{n\geq 1}$ be a sequence in $\mathbb{N}$. Let $\varepsilon > 0$ and n_0 be the least integer greater than $1/\varepsilon$. If $m, n > n_0$, then

$$d(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \max\left\{\frac{1}{m}, \frac{1}{n}\right\} < \frac{1}{n_0} < \varepsilon.$$

Thus, the sequence $\langle n \rangle_{n\geq 1}$ is Cauchy. Suppose contrary that the sequence $\langle n \rangle_{n\geq 1}$ converges to some point $p \in \mathbb{N}$. Let n_1 be any integer greater than $2p$. Then $n \geq n_1$ implies that

$$d(p, n) = \left|\frac{1}{p} - \frac{1}{n}\right| = \frac{1}{p} - \frac{1}{n} \geq \frac{1}{p} - \frac{1}{n_1} > \frac{1}{p} - \frac{1}{2p} = \frac{1}{2p}.$$

This shows that the sequence $\langle n \rangle_{n\geq 1}$ cannot converge to p, a contradiction to our supposition. Hence $(\mathbb{N}, d)$ is not a complete metric space.

Example 19 **Let $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}$ be defined by**

$$d(x, y) = \frac{|x - y|}{\sqrt{1 + x^2}\ \sqrt{1 + y^2}}, \text{ for all } x, y \in X.$$

Show that (X, d) is a metric space but not complete.

Solution:

(i) By the definition of absolute value of a real number, it follows that

$$d(x, y) = \frac{|x - y|}{\sqrt{1 + x^2}\ \sqrt{1 + y^2}} \geq 0; \qquad \forall x, y \in \mathbb{R}.$$

Thus d is non-negative.

(ii) Now $d(x, y) = 0 \Rightarrow |x - y| = 0$. A necessary and sufficient condition for $|x - y| = 0$ is that $x = y$. Thus

$$d(x, y) = 0 \Leftrightarrow x = y; \qquad \forall x, y \in \mathbb{R}.$$

(iii) Since $|p| = |-p|$, it follows that

$$\begin{aligned} d(x, y) &= \frac{|x - y|}{\sqrt{1 + x^2}\ \sqrt{1 + y^2}} = \frac{|-(y - x)|}{\sqrt{1 + x^2}\ \sqrt{1 + y^2}} \\ &= \frac{|y - x|}{\sqrt{1 + x^2}\ \sqrt{1 + y^2}} = d(y, x); \quad \forall x, y \in \mathbb{R}. \end{aligned}$$

Thus d is symmetric.

(iv) Finally, $\forall x, y, z \in \mathbb{R}$, we have

$$\begin{aligned} d(x, y) &= \frac{|x - y|}{\sqrt{1 + x^2}\ \sqrt{1 + y^2}} = \frac{|(x - z) + (z - y)|}{\sqrt{1 + x^2}\ \sqrt{1 + y^2}} \\ &\leq \frac{|x - z|}{\sqrt{1 + x^2}\ \sqrt{1 + z^2}} + \frac{|z - y|}{\sqrt{1 + z^2}\ \sqrt{1 + y^2}} \\ &= d(x, z) + d(z, y). \end{aligned}$$

So, d satisfies the triangle inequality.

Combining the above conditions, $(\mathbb{R}, d)$ is a metric space. Consider the sequence $\langle n \rangle_{n\geq 1}$ of natural numbers. Observe that

$$\begin{aligned} d(n,m) &= \frac{|n-m|}{\sqrt{1+n^2}\,\sqrt{1+m^2}} = \frac{|\frac{1}{n}-\frac{1}{m}|}{\sqrt{1+\frac{1}{n^2}}\,\sqrt{1+\frac{1}{m^2}}} \\ &\leq \left|\frac{1}{n}-\frac{1}{m}\right| \leq \frac{1}{m}+\frac{1}{n} \to 0 \text{ as } n,m\to\infty. \end{aligned}$$

Thus $\langle n \rangle_{n\geq 1}$ is a Cauchy sequence in (X, d). Suppose contrary that the sequence $\langle n \rangle_{n\geq 1}$ converges to some point $p \in \mathbb{R}$. Then

$$\begin{aligned} d(n,p) &= \frac{|n-p|}{\sqrt{1+n^2}\,\sqrt{1+p^2}} = \frac{|1-\frac{p}{n}|}{\sqrt{1+\frac{1}{n^2}}\,\sqrt{1+p^2}} \\ &\to \frac{1}{\sqrt{1+p^2}} \neq 0 \text{ as } n\to\infty. \end{aligned}$$

This shows that the sequence $\langle n \rangle_{n\geq 1}$ does not converge to $p \in \mathbb{R}$, a contradiction. Hence (X, d) is not a complete metric space.

6 Separable Metric Space

It is well known that the rational numbers or irrational numbers are densely packed along the real line. This phenomenon can be characterized interms of distance by saying that every point of $\mathbb{R}$ is zero distance from the set of rational numbers $\mathbb{Q}$ or from the set of irrational numbers $\mathbb{R}/\mathbb{Q}$. The concept of closure embled us to write this density property by the formula $\bar{\mathbb{Q}} = \mathbb{R}$ or $\overline{\mathbb{R}/\mathbb{Q}} = \mathbb{R}$. In this section, we extend the idea of density to an arbitrary metric space.

Definition 10 A metric space (X, d) is said to be *separable* if there exists a countable, everywhere dense set in X. Thus, $\exists$ a subset A of X such that

1. A is countable.
2. $\overline{A} = X$.

In other words, X is said to be separable if there exists in X a sequence $\langle x_1, x_2, \cdots \rangle$ such that for every $x \in X$, some sequence in the range of $\langle x_1, x_2, \cdots \rangle$ converges to x.

If the space is not separable, it is called *inseparable*. For example

(i) The real line usual metric space $\mathbb{R}_u$ is a separable metric space as $\mathbb{Q} \subset \mathbb{R}$ is a countable dense subset of $\mathbb{R}$. The set of rationals $\mathbb{Q}$ is a dense subset of $\mathbb{R}$ (usual metric) and so is the set of irrationals. Note that the former is countable whereas the latter is not.

(ii) A finite metric space is separable. Any countable metric space is separable.

(iii) The usual metric space $\mathbb{C}$ is separable, since the set $A = \{a + ib \in \mathbb{C} : a, b \in \mathbb{Q}\}$ is dense in $\mathbb{C}$.

(iv) The Euclidean space $\mathbb{R}^n$ is separable since the set $\mathbb{Q}^n = \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$ is countable and dense in $\mathbb{R}^n$.

(v) The metric space $\mathcal{C}[a,b]$ is separable since the set $\mathcal{P}[a,b]$ of all polynomials defined on $[a,b]$ with rational coefficients is countable and dense in $\mathcal{C}[a,b]$.

(vi) Let (X,d) be a discrete metric space. Since every subset is closed, the only dense subset is X itself, so, X is countable and as such the space is separable.

(vii) Let X be the uncountable set and d be the discrete metric on X then (X,d) is not separable as no proper subset of X in the discrete metric space (X,d) can be dense subset of X.

(viii) Consider the set A of elements $x = \{x_1, x_2, \cdots\}$ of X for which each x_i is either 0 or 1. Let E be any countable subset of A, then the elements of E can be arranged in a sequence $s_1, s_2, \cdots$. We construct a sequence s as follows: If the m^{th} element of s_m is 1, then the m^{th} element of s is 0, and vice versa. Then the element s of X differs from each s_m in the m^{th} place and is therefore equal to none of them. So, $s \notin E$ although $s \in A$. This shows that any countable subset of A must be a proper subset of A. It follows that A is uncountable, for if it were to be countable, then it would have to be a proper subset of itself, which is absurd. We proceed to use the uncountability of the subset A to argue that X must be inseparable.

(ix) The distance between two distinct elements $x = \{x_1, x_2, \cdots\}$ and $y = \{y_1, y_2, \cdots\}$ of A is

$$d(x,y) = \sup\{|x_i - y_i| : i = 1,2,3,\cdots\} = 1.$$

Suppose, if possible, that E_0 is a countable, everywhere dense subset of X. Consider the balls of radii $1/3$ whose centres are the points of E_0. Their union is the entire space X, because E_0 is everywhere dense, and in particular contains A. Since the balls are countable in number while A is not, in at least one ball there must be two distinct elements x and y of A. Let x_0 denote the centre of such a ball. Then

$$1 = d(x,y) \le d(x,x_0) + d(x_0,y) < \frac{1}{3} + \frac{1}{3} < 1,$$

which is, however, impossible. Consequently, (X,d) cannot be separable.

(x) Consider l_∞ space. Consider a subset $A = \{x = \langle x_n \rangle \subseteq l_\infty : x_n = 0 \text{ or } 1\}$ of l_∞. With x we associate a real number x^* such that the binary expansion of x^* is

$$x^* = 0.x_1x_2x_3\cdots \in [0,1].$$

Since each real number in $[0,1]$ has a unique binary expansion, distinct real numbers in $[0,1]$ give rise to distinct sequences of zeros and ones. Since $[0,1]$ is uncountable, the set of sequences consisting of zeros and ones in l_∞ is uncountable. Also, the metric of l_∞ shows that the distance between two distinct elements x and y of A is $d(x,y) = \sup\limits_{1\le n<\infty} |x_n - y_n| = 1$.. If we let each of these sequences to be the centre x_0 of an open sphere of radius $1/3$, then

$$1 = d(x,y) \le d(x,x_0) + d(x_0,y) < \frac{1}{3} + \frac{1}{3} < 1$$

and there will be uncountable non-intersecting open spheres. If E be the dense set in l_∞, then it will be intersecting with each of these open spheres. Therefore, E is uncountable. Since E is arbitrary set, l_∞ can not be separable.

Theorem **27** **Let (X, d) be a metric space and $Y \subset X$. If X is separable, then Y with the induced metric is separable, too.**

Proof: Let $E = \{x_i : i = 1, 2, \cdots\}$ be a countable dense subset of X. If E is contained in Y, then there is nothing to prove. Otherwise, we construct a countable dense subset of Y whose points are arbitrarily close to those of E. For $n, m \in \mathbb{N}$, let $S_{n,m} = S(x_n, 1/m)$ and choose $y_{n,m} \in S_{n,m} \cap Y$ whenever this set is nonempty. We show that the countable set $\{y_{n,m} : n$ and m positive integers$\}$ of Y is dense in Y.

For this purpose, let $y \in Y$ and $\varepsilon > 0$. Let m be so large that $1/m < \varepsilon/2$ and find $x_n \in S(y, 1/m)$. Then $y \in S_{n,m} \cap Y$ and

$$d(y, y_{n,m}) \leq d(x, x_n) + d(x_n, y_{n,m}) < \frac{1}{m} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $y_{n,m} \in S(y, \varepsilon)$. Since $y \in Y$ and $\varepsilon > 0$ are arbitrary, the assertion is proved.

Theorem **28** **A metric space (X, d) is separable if and only if it is second countable.**

Proof: Suppose (X, d) is separable. Then X has a countable dense sub-set A (say). Let $A = \{a_n : n \in \mathbb{N}\}$, so $\overline{A} = X$.

Let us construct $B = \left\{ S_{\frac{1}{n}}(a_i) : a_i \in A,\ n \in \mathbb{N} \right\}$. Here B is a countable collection of countable set. So B is countable. We now prove that B is a base of (X, d).

Let G be any open set in (X, d) and $x \in G$. Since G is open, x is an interior point of G. So, $\exists\ r > 0$ such that $S_r(x) \subseteq G$. Now by Archimidian property $\exists$ a natural number m such that $\frac{1}{n} < \frac{r}{2}$. Since

$$x \in X = \overline{A},\ S_{\frac{1}{m}} \cap A \neq \phi$$

so $\exists\ a_i \in A$ such that $a_i \in S_{\frac{1}{m}}(x)$, i.e., $d(x, a_i) < \frac{1}{m}$. Let $y \in S_{\frac{1}{m}}(a_i)$. Then $d(y, a_i) < \frac{1}{m}$. Now

$$\begin{aligned} d(x, y) &\leq d(x, a_i) + d(y, a_i) < \frac{1}{m} + \frac{1}{m} = \frac{2}{m} \\ &< r;\ \text{as } \frac{1}{m} < \frac{r}{2} \\ \text{i.e.,} \quad & y \in S_{\frac{1}{m}}(a_i) \quad \Rightarrow \quad y \in S_r(x). \end{aligned}$$

So, $S_{\frac{1}{m}}(a_i) \subseteq S_r(x) \subseteq G$, i.e., $x \in S_{\frac{1}{m}}(a_i) \subseteq G$. So, B is a base of (X, d). Hence (X, d) is second countable.

Conversely suppose (X, d) is second countable and $B = \{B_n : n \in \mathbb{N}\}$, be a countable base of (X, d). We take a point $b_n \in B_n$ for each $n \in \mathbb{N}$ and construct $A = \{b_n : n \in \mathbb{N}\}$. So, A is a countable sub-set of X. We shall prove that $\overline{A} = X$.

Let $x \in X$ and $r > 0$. Since B is a base, $\exists\ B_k \in B$ such that, $x \in B_k \subseteq S_r(x)$. Now

$$\begin{aligned} & b_k \in B_k \subseteq S_r(x) \text{ and } b_k \in A \\ \text{so} \quad & b_k \in S_r(x) \cap A \Rightarrow \quad S_r(x) \cap A \neq \phi \end{aligned}$$

Since $r > 0$ is arbitrary, $S_r(x) \cap A \neq \phi$ for each $r > 0 \Rightarrow x \in \overline{A}$. So $x \subseteq \overline{A}$ consequently $\overline{A} = X$. Hence (X, d) is separable.

Example **20** **Prove that the space $l_p,\ 1 \le p < \infty$ is separable.**

Solution: Let $l_p(p \ge 1)$ consists of all those sequences $x = \langle x_n \rangle$ of real or complex numbers, such that $\sum_{n=1}^{\infty} |x_n|^p$ is convergent. We define a function $d : l_p \times l_p \longrightarrow \mathbb{R}$ given by

$$d(x, y) = \Big[\sum_{n=1}^{\infty} \Big|x_n - y_n\Big|^p\Big]^{1/p}; \quad x = \langle x_n \rangle, y = \langle y_n \rangle \in l_p.$$

Let us construct a set A containing those elements of l_p, the first n components of which are rational numbers and the rest are all zeros, i.e.,

$$A = \Big\{(\beta_1, \beta_2, \cdots \beta_n, 0, 0, \cdots); \beta_i \in \mathbb{Q}, 1 \le i \le n\Big\}.$$

Then $A = \bigcup_{k=1}^{\infty} A_k$, where for $k \in \mathbb{N}$,

$$A_k = \Big\{(\beta_1, \beta_2, \cdots \beta_k, 0, 0, \cdots); \beta_i \in \mathbb{Q}, 1 \le i \le k\Big\}.$$

Each A_k is countable. For $(\beta_1, \beta_2, \cdots \beta_n) \in \mathbb{Q}^n$, the mapping

$$(\beta_1, \beta_2, \cdots \beta_n) \to (\beta_1, \beta_2, \cdots \beta_n, 0, 0, \cdots)$$

is a one-to-one correspondence between $\mathbb{Q}^n$ and A_n, and $\mathbb{Q}^n$ is a countable set. Thus, A, being a countable union of countable sets, is a countable set.

Now, we show that A is dense in l_p. Let $x = \langle \alpha_i \rangle$ be an arbitrary element of l_p. Then $\sum_{n=1}^{\infty} \big|\alpha_n\big|^p$ is convergent, and so corresponding to an $\varepsilon > 0$, $\exists m \in \mathbb{N}$, such that

$$\sum_{i=m+1}^{\infty} \big|\alpha_i\big|^p < \frac{\varepsilon^p}{2}.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for each real number $\alpha_i (i = 1, 2, \cdots, m)$ $\exists$ a natural number β_i such that

$$\sum_{i=1}^{m} \big|\alpha_i - \beta_i\big|^p < \frac{\varepsilon^p}{2}.$$

Let $y = \{\beta_1, \beta_2, \cdots \beta_n, 0, 0, \cdots\}$ then $y \in A$. Now

$$\begin{aligned} d(x, y) &= \Big\{\sum_{i=1}^{\infty} \big|\alpha_i - \beta_i\big|^p\Big\}^{\frac{1}{p}} \\ &= \Big\{\sum_{i=1}^{m} \big|\alpha_i - \beta_i\big|^p + \sum_{i=m+1}^{\infty} \big|\alpha_i\big|^p\Big\}^{\frac{1}{p}} \\ &< \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} = \varepsilon \end{aligned}$$

So, $y \in S_\varepsilon(x)$. Since $\varepsilon > 0$ is arbitrary, $S_\varepsilon(x) \cap A \ne \phi$ for each $\varepsilon > 0$.

$\therefore x \in \overline{A}$. So $l^p \subseteq \overline{A} \Rightarrow \overline{A} = l^p$. Hence l^p is separable.

Example **21** **Let (X, d) be a metric space and Y be a subset of X. If Y is separable and $\bar{Y} = X$, then prove that X is separable.**

Solution: Let A be a countable dense subset of Y. Let $x \in X$ and $\varepsilon > 0$ be given arbitrary. Since $\bar{Y} = X$, there exists $y \in Y$ such that $d(x, y) < \varepsilon/2$. Also, since A is dense in Y, there exists $z \in A$ such that $d(y, z) < \varepsilon/2$. By the triangle inequality, we have

$$d(x, z) \leq d(x, y) + d(y, z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It follows that A is dense in X and hence X is separable.