Complex Integration

1 Integrals of a Function

The derivative of a complex-valued complex function $w(t) = u(t) + iv(t)$, where $t \in \mathbb{R}$ is

$$w'(t) = \frac{d}{dt}w(t) = u'(t) + i\,v'(t).$$

Thus, for every complex constant $z_0 = x_0 + iy_0$,

$$\begin{aligned} \frac{d}{dt}\Big[z_0\,w(t)\Big] &= \frac{d}{dt}\Big[(x_0 + iy_0)(u + iv)\Big] = (x_0u' - y_0v') + i(y_0u' + x_0v') \\ &= (x_0 + iy_0)(u' + iv') = z_0\,w'(t). \end{aligned}$$

1.1 Definite Integrals of Functions

Let $w(t) = u(t) + iv(t)$ be a complex-valued function of a real variable t, defined on $[a, b]$, where $u(t)$ and $v(t)$ are real-valued functions of real variable t. Then , the *definite integral* of complex valued function $w(t)$ over the real interval $[a, b]$ is defined as

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt, \tag{1}$$

provided the functions $u(t)$ and $v(t)$ are integrable in $[a, b]$. The integrals of $u(t)$ and $v(t)$ in Eq. (1) exist, if these functions are piecewise continuous over $[a, b]$.

2 Contours

2.1 Path / Arc

(i) A *path* or *arc* is the set of all image points of a closed finite interval under a continuous mapping. Thus *path* in a plane $\mathbb{C}$ from A to B is a continuous function $t \to \gamma(t)$ on some parameter interval $a \le t \le b$ such that $\gamma(a) = A$ and $\gamma(b) = B$.

(ii) The path is *simple* , if $\gamma(s) \ne \gamma(t)$, when $s \ne t$. The path is *closed,* if it starts and ends at the same point, i.e., $\gamma(a) = \gamma(b)$. The *simple closed path* is a closed path γ such that $\gamma(s) \ne \gamma(t)$, when $a \le s < t < b$. The *trace* of the path γ is its image $\Gamma = \gamma([a, b])$ of γ, which is a subset of the plane. For example, the parameterization of a unit circle is $\gamma(t) = \cos t + i\sin t$ or $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$, which represents a closed path.

(iii) A *smooth path* is a path that can be represented in the form $\gamma(t) = x(t) + i\, y(t);\ a \leq t \leq b$, where the functions $x(t)$ and $y(t)$ are smooth (i.e., they have many derivatives). In other words, a differentiable arc is called *smooth arc* if $\gamma'(t) \neq 0$ for all $t \in (a, b)$. The point $\gamma(a) = x(a)+iy(a)$ or $A = (x(a), y(a))$ is called the *initial point* of $\mathcal{C}$ and $\gamma(b) = x(b) + iy(b)$ or $B(x(b), y(b))$ is its *terminal point*.

If an arc $\mathcal{C}$ is given by $\gamma(t) : t \in [a, b]$, then the opposite curve $-\mathcal{C}$ is defined by $\phi(t) = \gamma(a + b - t) : t \in [a, b]$.

2.2 Contour/ Piecewise Smooth Arc

A contour, or a piecewise smooth arc or a sectionally smooth arc is finite number of smooth arcs joined end to end. A curve Γ with a parametric equation $\gamma(t), t \in [a, b]$ is called *piecewise smooth curve* if $\gamma(t)$ is differentiable and $\gamma'(t)$ is continuous for all but finite number of points $\{t_j : 1 \leq j \leq n\}$. Furthermore, $\gamma(t)$ has both left limit and right limit at each $t_j; 1 \leq j \leq n$.

A *contour* is defined to be *closed* if $\gamma(a) = \gamma(b)$ only. A contour is called a *simple closed contour* or *Jordan arc* if and only if there is no self-intersection except that the initial point equals the final point. For example

(i) the boundaries of triangles, rectangles and squares is a contour.

(ii) $z = e^{it}, 0 \leq t \leq 2\pi$ is a closed contour.

Suppose $f(z)$ is continuous on Γ (we mean now Γ regarded as a set of points $\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n$ in the plane, namely the union of the sets Γ_j and these are in turn the ranges of the parametric curves Γ_j). Then we define the integral of f along the chain Γ as

$$\int_\Gamma f(z)\, dz = \sum_{j=1}^{n} \int_{\Gamma_j} f(z)\, dz.$$

Let $\mathcal{D}$ be a region and Γ be piecewise smooth curve in $\mathcal{D}$. If f is a continuous function on $\mathcal{D}$, then define the *line integral*

$$\int_\Gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt, \tag{2}$$

where $z = \gamma(t); t \in [a, b]$ is a parametric equation of Γ.

2.3 Rectifiable Arcs

Let an arc Γ be defined by $\gamma(t) = x(t) + iy(t),\ t \in [a, b]$. Consider a partition $P = \{t_0, t_1, \cdots, t_n\}$ of $[a, b]$. Corresponding to this partition P, Γ is divided into n subarcs (Fig. 1) $\Gamma_k = arc\ z_{k-1}z_k$; $k = 1, 2, \cdots, n$, where $z_k = \gamma(t_k); k = 0, 1, 2, \cdots, n$. We join each of the points $z_0, z_1, \cdots, z_n$ to the next point by straight lines. In this way, we obtain a polygonal curve. The length of this polygonal curve is evidently $\sum_{k=1}^{n} |z_k - z_{k-1}|$. The *length of the arc* is defined by a nonnegative real number

$$L = \sup_P \sum_{k=1}^{n} |z_k - z_{k-1}| = \sup_P \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})|. \tag{3}$$

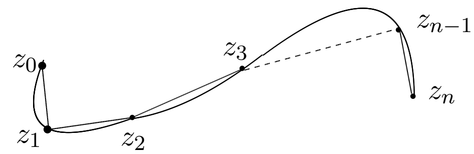

Figure 1: Rectifiable arcs.

A curve of finite length i.e., $L < \infty$, is called a *rectifiable curve*.

Let a contour $\mathcal{C}$ be parametrised by $z = \gamma(t), a \leq t \leq b$, with $\gamma'(t)$ exists and is a continuous function for all t satisfying $a \leq t \leq b$. Then, the length L of contour $\mathcal{C}$ is given by

$$L = \int_a^b |\gamma'(t)| dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} \; dt. \tag{4}$$

It is evident that a contour is rectifiable; its length is the sum of lengths of smooth arcs comprising the contour.

Definition 1. A region determined by a closed curve $\mathcal{C}$ is defined by a component of the complement of $\mathcal{C}$ in the extended complex plane. This definition can be understood by the Fig. 2. For a given closed

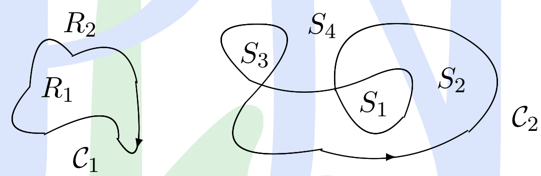

Figure 2: Region determined by a closed curve.

curve, among the regions determined by $\mathcal{C}$, these must be only one unbounded region determined by $\mathcal{C}$. That region is treated as the region containing ∞ in $\mathbb{C}_\infty$. In Fig. 2, R_2 and S_4 are the unbounded regions determined by $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively.

Definition 2. (**Connected region**): A region is said to be *connected* if any two points of the region can be joined by an arc, then the arc lies wholly within the region, that is, every point of the arc belongs to the region, otherwise it is *disconnected*.

(i) (**Simply connected region**) : A region $\mathcal{R}$ in $\mathbb{C}$ is said to be a *simply connected region* , (Fig. 3), if

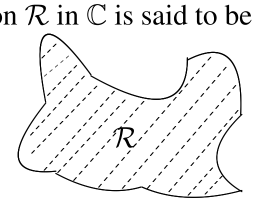

Figure 3: A simply connected region.

any closed curve lying entirely in $\mathcal{R}$ can be contracted continuously to a point without any portion of the curve passing out of $\mathcal{R}$ in that process of contraction. For example, a circular region, a cubical region, a spherical region are examples of simply connected region.

(ii) (**Multiply connected region**) : A region $\mathcal{R}$ in $\mathbb{C}$ which is not simply connected, is called a *multiply connected region*, Fig. 4. That is, the complement of $\mathcal{R}$ in $\mathbb{C}_\infty$ has more than two components or equivalently; there exists a closed curve $\mathcal{C}$ in $\mathcal{R}$, which encloses a point of complement of $\mathcal{R}$. For example, the regions bounded by two concentric circles, spheres are multiply connected regions.

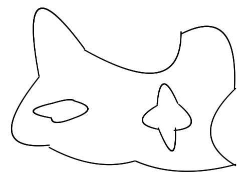

Figure 4: A multiply connected region.

RESULT **1. Jordan curve theorem :** The complement of a simple closed curve has exactly two regions.

3 Complex Line Integration

An integral of a function f of a complex variable z that is defined on a contour $\mathcal{C}$ is denoted by $\int_{\mathcal{C}} f(z)\, dz$ and is called a *complex integral.* Let Γ be a rectifiable path. Let $f(z)$ be a complex valued continuous function of complex variable z, defined at all points on the smooth curve Γ that lies in some region of

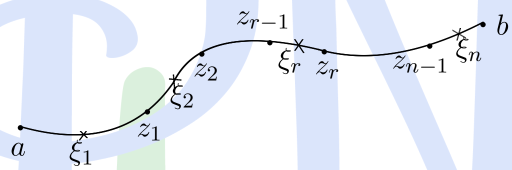

Figure 5: Complex line integrals

the complex plane. Let the partition of Γ by means of points $z_1, z_2, \cdots, z_{n-1}$, chosen arbitrarily and call $a = z_0, b = z_n$ (Fig. 5). Let the length of the subinterval be

$$\Delta z_r = z_r - z_{r-1}; \quad r = 1, 2, \cdots, n.$$

On each arc joining z_{r-1} and z_r (where r goes from 1 to n), choose a point ξ_r. Consider the sum

$$\begin{aligned} S_n &= (z_1 - a)f(\xi_1) + (z_2 - z_1)f(\xi_2) + \cdots + (b - z_{n-1})f(\xi_n) \\ &= \sum_{r=1}^{n}(z_r - z_{r-1})f(\xi_r) = \sum_{r=1}^{n} f(\xi_r)\Delta z_r \;;\; \xi_r \in (z_{r-1}, z_r). \end{aligned} \tag{5}$$

Let the number of subdivisions n increase in such a way that the largest of chord lengths $|\Delta z_r|$ (i.e., norm) approaches zero. Then, if the sum S_n approaches to a definite limit, irrespective of the more of partitions of Γ and the choice of ξ_r's, then the limit is said to be complex integral of $f(z)$ on Γ from $z_0(= a)$ to $z_n(= b)$, written as

$$\int_a^b f(z)dz \text{ or } \int_\Gamma f(z)dz = \lim_{n\to\infty} \sum_{r=1}^{n} f(\xi_r)\Delta z_r. \tag{6}$$

Eq.(6) is called the complex line integral or contour integral or briefly line integral of $f(z)$ along curve Γ, or the definite integral of $f(z)$ from a to b along curve Γ.

EXAMPLE **1.** *Let Γ be the circle with centre a and radius r, prove that*

$$\int_\Gamma \frac{dz}{(z-a)^n} = \begin{cases} 0, & \text{if } n \neq 1 \\ 2\pi i, & \text{if } n = 1 \end{cases}, \quad n \in \mathbb{Z}.$$

Solution: The parametric equation of the circle Γ is given by $z - a = re^{it}$, $0 \le t \le 2\pi$, then

$$\begin{aligned}\int_\Gamma \frac{dz}{(z-a)^n} &= \int_0^{2\pi} \frac{i\,r\,e^{it}}{(re^{it})^n}\,dt = \frac{i}{r^{n-1}}\int_0^{2\pi} e^{i(1-n)t}\,dt \\ &= \frac{i}{r^{n-1}}\left[\frac{e^{i(1-n)t}}{i(1-n)}\right]_0^{2\pi};\quad \text{provided } n \neq 1 \\ &= \frac{1}{(1-n)r^{n-1}}\left[e^{i(1-n)2\pi} - 1\right] \\ &= \frac{1}{(1-n)r^{n-1}}\left[1-1\right] = 0, \text{ if } n \neq 1.\end{aligned}$$

In particular, if $n = 1$, then

$$\int_\Gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{i\,r\,e^{it}}{re^{it}}\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

Hence the result.

EXAMPLE **2.** *Let Γ be the semi circular path $z = 3e^{i\theta}$ $(0 \le \theta \le \pi)$ and the branch of the function $z^{1/2}$ be $f(z) = z^{1/2} = e^{\log z/2}$ $(|z| > 0, 0 < \arg z < 2\pi)$. Then evaluate the integral $I = \int_\Gamma z^{1/2}\,dz$.*

Solution: Given, Γ is the semi circular path $z = 3e^{i\theta}$ $(0 \le \theta \le \pi)$ from the point $z = 3$ to the point $z = -3$ (Fig. 6). Although the branch $f(z) = z^{1/2} = e^{\log z/2}$ $(|z| > 0, 0 < \arg z < 2\pi)$ of the

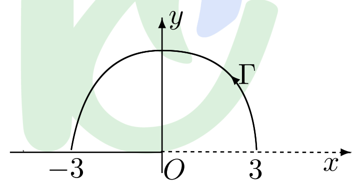

Figure 6: Branch cut and contour integrals

multi-valued function $z^{1/2}$ is not defined at the initial point $z = 3$ on Γ, the integral I nevertheless exists. This is because when $z(\theta) = 3e^{i\theta}, 0 \le \theta \le \pi$, we have

$$f[z(\theta)] = \exp\left[\frac{1}{2}(\ln 3 + i\theta)\right] = \sqrt{3}\,e^{i\theta/2}$$

which is piecewise continuous on $[0, \pi]$. Hence, the right had limits of the real and imaginary components of the function

$$f[z(\theta)]z'(\theta) = \sqrt{3}\,e^{i\theta/2}\,3ie^{i\theta} = 3\sqrt{3}\,e^{i3\theta/2} = -3\sqrt{3}\sin\frac{3\theta}{2} + i3\sqrt{3}\cos\frac{3\theta}{2}$$

where $0 \le \theta \le \pi$ exist at $\theta = 0$ and these limits are 0 and $3\sqrt{3}$, respectively. Thus $f[z(\theta)]z'(\theta)$ is continuous on $[0, \pi]$ when its value is $i3\sqrt{3}$ at $\theta = 0$.

$$\therefore I = i3\sqrt{3}\int_0^\pi e^{i3\theta/2}\,d\theta = -2\sqrt{3}(1+i).$$

EXAMPLE **3.** *Evaluate $I = \int_\mathcal{C} \bar{z}^2\,dz + z^2\,d\bar{z}$ along the curve $\mathcal{C}$ defined by $z^2 + 2z\bar{z} + \bar{z}^2 = (2-2i)z + (2+2i)\bar{z}$ from the point $z = 1$ to $z = 2 + 2i$.*

Solution: Let $z = x + iy$, the equation of the given curve can be written as

$$\begin{aligned} & (z+\bar{z})^2 = 2(z+\bar{z}) - 2i(z-\bar{z}) \\ \Rightarrow \quad & (2x)^2 = 2\cdot 2x - 2i\cdot 2iy \Rightarrow y = x(x-1), \end{aligned}$$

The parametric equation of the curve is $x = t, y = 2t^2 - t$, where $1 \le t \le 2$. Therefore

$$\begin{aligned} I &= \int_{z=1}^{2+2i} \Big[(x-iy)^2(dx+idy) + (x-iy)^2(dx-idy)\Big] \\ &= \int_{z=1}^{2+2i} \Big[\Big\{(x-iy)^2 + (x+iy)^2\Big\}dx + i\Big\{(x-iy)^2 - (x+iy)^2\Big\}dy\Big] \\ &= 2\int_{(1,0)}^{(2,2)} \Big[(x^2-y^2)dx + 2xy\,dy\Big] \\ &= 2\int_1^2 (3t^4 - 8t^3 + 5t^2)dt = 248/15. \end{aligned}$$

3.1 Estimation of Contour Integrals

THEOREM **1.** *Let $f(t)$ is a complex-valued function integrable over $[a,b]$. Then*

$$\Big|\int_a^b f(t)\,dt\Big| \le \int_a^b |f(t)|\,dt. \tag{7}$$

Proof: If $a = b$ or $\int_a^b f(t)\,dt = 0$, then the above inequality holds. Let $a < b$ and the value of the integral is a non-zero complex number $re^{i\theta}$. Therefore

$$\int_a^b f(t)\,dt = r\,e^{i\theta} \Rightarrow r = e^{-i\theta}\int_a^b f(t)\,dt = \int_a^b e^{-i\theta}\,f(t)\,dt.$$

Also, $r = |re^{i\theta}| = \Big|\int_a^b f(t)\,dt\Big|$. Since r is a positive real number, so the integral on the right hand side is also a positive real number. We know that the real part of a real number is the number itself. Therefore

$$\begin{aligned} r &= \mathcal{R}e\int_a^b e^{-i\theta}\,f(t)\,dt = \int_a^b \mathcal{R}e\Big[e^{-i\theta}\,f(t)\Big]\,dt \\ &\le \int_a^b \Big|e^{-i\theta}\,f(t)\Big|\,dt, \quad \text{as } \mathcal{R}e\,z \le |z| \\ &\le \int_a^b \Big|e^{-i\theta}\Big|\,|f(t)|\,dt = \int_a^b \Big|f(t)\Big|\,dt \\ \therefore \quad & \Big|\int_a^b f(t)\,dt\Big| \le \int_a^b \Big|f(t)\Big|\,dt. \end{aligned}$$

THEOREM **2** (The absolute value of a complex integral)**.** *If a function $f(z)$ is piecewise continuous on a contour Γ of finite length L, and if $M \in \mathbb{R}^+$ be the upper bound of $|f(z)|$ on Γ, then*

$$\Big|\int_\Gamma f(z)\,dz\Big| \le M\cdot L. \tag{8}$$

This is the upper bound for moduli of contour integrals.

Proof. Let $f(z)$ be continuous at all points of the rectifiable curve Γ having length L. Subdivide Γ into n parts by means of points $z_1, z_2, \cdots, z_{n-1}$, chosen arbitrarily, and call $a = z_0, b = z_n$, where $L = b - a$. On the arc joining z_{k-1} to z_k (where k goes from 1 to n), choose a point ξ_k. Then by definition (6), we have

$$\int_\Gamma f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta z_k.$$

Now, using the property of modulus, we have,

$$\Big|\sum_{k=1}^{n} f(\xi_k)\Delta z_k\Big| \le \sum_{k=1}^{n} |f(\xi_k)|\cdot|\Delta z_k| \le M\sum_{k=1}^{n}|\Delta z_k| = ML, \tag{9}$$

where we have used the facts that $|f(z)| \le M$ for all points z on Γ and that $\sum\limits_{k=1}^{n}|\Delta z_k|$ represents the sum of all the chord lengths joining points z_{k-1} and z_k, where $k = 1, 2, \cdots, n$ and that this sum is not greater than the length of Γ. Since all of the paths of integration to be considered here are contours and the integrals are piecewise continuous functions defined on those contours, a number M will always exist. Taking limits on both sides of (9), we get,

$$\lim_{n\to\infty}\Big|\sum_{k=1}^{n} f(\xi_k)\Delta z_k\Big| \le ML$$

$$\text{or,}\quad \Big|\int_\Gamma f(z)dz\Big| \le M\,L.$$

It is sometimes referred to as ML-inequality.

EXAMPLE **4.** *Find the upper bound for the absolute value of* $\displaystyle\int_{\mathcal{C}} \frac{(z^2+3)e^{iz}\,Log\,z}{z^2-2}\,dz$, *where* $\mathcal{C} = \Big\{z : z = 2e^{i\theta}, 0 \le \theta \le \pi/3\Big\}$.

Solution: Let $f(z) = \{(z^2+3)e^{iz}\text{ Log } z\}/(z^2-2)$. The contour (Fig. 7) is the part of the circular arc

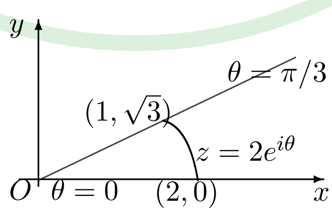

Figure 7: Complex line integrals

$\mathcal{C} = \Big\{z : z = 2e^{i\theta}, 0 \le \theta \le \pi/3\Big\}$. Then

$$\begin{aligned}
|f(z)| &= \Big|\frac{(z^2+3)e^{iz}\text{ Log } z}{z^2-2}\Big| \le \frac{(|z|^2+3)e^{-y}|e^{ix}||\ln r + i\text{ Arg } z|}{||z|^2-2|}\\
&\le \frac{(|z|^2+3)e^{-y}|\ln r + i\text{ Arg } z|}{||z|^2-2|}\\
&\le \frac{(2^2+3)\cdot 1\cdot \ln(2+\pi/3)}{|2^2-2|} = \frac{7(\ln 2+\pi/3)}{2},\quad \text{as } e^{-y}\le 1 \text{ for } 0\le y\le\sqrt{3}\\
&\le \frac{7}{6}(3\ln 2+\pi) = M.
\end{aligned}$$

Also, $L = \int_0^{\pi/3} 2\, d\theta = 2\pi/3$. Hence by using Eq. (8), we get

$$\Big| \int_{\mathcal{C}} f(z)dz \Big| \leq M\, L = \frac{7\pi}{9}(3\ln 2 + \pi).$$

4 Cauchy's Theorem

THEOREM **3.** *If $f(z)$ is a single valued, analytic and regular in a simply connected region $\mathcal{D}$, and $f'(z)$ is continuous in $\mathcal{D}$. Then, for every simple closed contour Γ in $\mathcal{D}$, $\int_\Gamma f(z)dz = 0$.*

Proof: Let $f(z) = u(x, y) + iv(x, y)$ and $z = x + iy$; x, y, u, v being real-valued. Then

$$f(z)dz = (u + iv)(dx + idy) = (u\, dx - v\, dy) + i(v\, dx + u\, dy).$$

Also, let $\mathcal{D} =$ Int Γ, the interior of Γ. The integral of f over Γ can be written as

$$\int_\Gamma f(z)\, dz = \int_\Gamma (u\, dx - v\, dy) + i\int_\Gamma (v\, dx + u\, dy). \tag{10}$$

Since f is analytic in $\mathcal{D}$ (and hence continuous in $\mathcal{D}$), u and v are also continuous therein and

$$f'(z) = u_x + iv_x = v_y - iu_y$$

so that $u_x = v_y$ and $u_y = -v_x$. Further, as f' is continuous in $\mathcal{D}$, the partial derivatives of u and v are also continuous in $\mathcal{D}$. By applying Green's theorem, to each of the integrals on the right hand side of Eq. (10), we obtain

$$\begin{aligned} \int_\Gamma f(z)\, dz &= \int_\Gamma (u\, dx - v\, dy) + i\int_\Gamma (v\, dx + u\, dy) \\ &= \int\int_{\mathcal{D}} \Big(- v_x - u_y\Big) dx\, dy + i\int\int_{\mathcal{D}} \Big(u_x - v_y\Big) dx\, dy \\ &= 0 \end{aligned}$$

which is the well-known Cauchy's theorem.

4.1 Cauchy-Goursat Theorem

THEOREM **4.** *If f is analytic at all points within and on a simple closed contour Γ, then $\oint_\Gamma f(z)\, dz = 0$.*

Proof: Proof of the theorem for the case of a triangle: Consider any triangular contour Δ in the z-plane contained in $\mathcal{D}$ (Fig. 8). Since $\mathcal{D}$ is simply connected, the interior of $\triangle$ also belong to $\mathcal{D}$. To achieve the desired result, join the midpoints D, E and F of the sides AB, AC and BC respectively to form four triangles indicated briefly by $\triangle_I, \triangle_{II}, \triangle_{III}$ and $\triangle_{IV}$. If $f(z)$ is analytic inside and on triangle ABC,

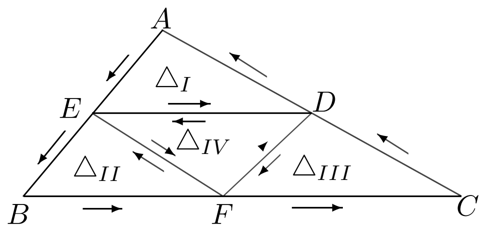

Figure 8: Triangular region

we have,

$$\begin{aligned}
\oint_{ABCA} f(z)dz &= \int_{DAE} f(z)dz + \int_{EBF} f(z)dz + \int_{FCD} f(z)dz \\
&= \Big[\{\int_{DAE} + \int_{ED}\} + \{\int_{EBF} + \int_{FE}\} + \{\int_{FCD} + \int_{DF}\} + \{\int_{DE} + \int_{EF} + \int_{FD}\}\Big] f(z)dz \\
&= \int_{DAED} f(z)dz + \int_{EBFE} f(z)dz + \int_{FCDF} f(z)dz + \int_{DEFD} f(z)dz \\
&= \oint_{\triangle_I} f(z)dz + \oint_{\triangle_{II}} f(z)dz + \oint_{\triangle_{III}} f(z)dz + \oint_{\triangle_{IV}} f(z)dz \\
\text{or} \Big|\oint_{\triangle} f(z)dz\Big| &\leq \Big|\oint_{\triangle_I} f(z)dz\Big| + \Big|\oint_{\triangle_{II}} f(z)dz\Big| + \Big|\oint_{\triangle_{III}} f(z)dz\Big| + \Big|\oint_{\triangle_{IV}} f(z)dz\Big|. \qquad (11)
\end{aligned}$$

Let $\triangle_1$ be the triangle corresponding to that term on the right hand side of Eq. (11) having largest value (if there are two or more such terms then $\triangle_1$ is any of the associated triangle). Then

$$\Big|\oint_{\triangle} f(z)dz\Big| \leq 4\Big|\oint_{\triangle_1} f(z)dz\Big|. \qquad (12)$$

By joining midpoints of the sides of triangle $\triangle_1$, we obtain similarly a triangle $\triangle_2$ such that

$$\begin{aligned}
\Big|\oint_{\triangle_1} f(z)dz\Big| &\leq 4\Big|\oint_{\triangle_2} f(z)dz\Big| \\
\Re\Big|\oint_{\triangle} f(z)dz\Big| &\leq 4^2\Big|\oint_{\triangle_2} f(z)dz\Big|. \text{ by Eq. 12} \qquad (13)
\end{aligned}$$

Proceeding in this way, after n steps we obtain a triangle $\triangle_n$ such that

$$\Big|\oint_{\triangle} f(z)dz\Big| \leq 4^n\Big|\oint_{\triangle_n} f(z)dz\Big|. \qquad (14)$$

Now, $\triangle, \triangle_1, \triangle_2, \cdots$ is a sequence of nested triangles each of which is contained in the preceding (i.e., a sequence of nested triangles) and there exists a point z_0 which lies in every triangle of the sequence. Since z_0 lies inside or on the boundary of $\triangle$, it follows that, $f(z)$ is analytic at z_0, then

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + (z - z_0)\eta \qquad (15)$$

where for any $\varepsilon > 0$, we can find δ such that $|\eta| < \varepsilon$, whenever $|z - z_0| < \delta$. Thus, by integration of both sides of (15) and using Cauchy's theorem 3, we obtain

$$\begin{aligned}\oint_{\triangle_n} f(z)dz &= \oint_{\triangle_n} f(z_0)dz + \oint_{\triangle_n} (z - z_0)f'(z_0)dz + \oint_{\triangle_n} (z - z_0)\eta dz \\ &= f(z_0)\oint_{\triangle_n} dz + f'(z_0)\oint_{\triangle_n} (z - z_0)dz + \oint_{\triangle_n} (z - z_0)\eta dz \\ &= \oint_{\triangle_n} (z - z_0)\eta dz\ ; \quad \text{as } \oint_{\triangle_n} dz = 0 = \oint_{\triangle_n} (z - z_0)dz. \qquad (16)\end{aligned}$$

Now, let P be the perimeter of $\triangle$, then the perimeter of $\triangle_n$ is $P_n = \frac{1}{2^n}P$. If z is any point in $\triangle_n$, then we must have, $|z - z_0| < \frac{P}{2^n} < \delta$. Hence

$$\left|\oint_{\triangle_n} f(z)dz\right| = \left|\oint_{\triangle_n} (z - z_0)\eta dz\right| \le \varepsilon\frac{P}{2^n}\cdot\frac{P}{2^n} = \frac{\varepsilon P^2}{4^n}$$

$$\Re \quad \left|\oint_{\triangle} f(z)dz\right| \le \varepsilon P^2\ ; \text{ from (14).} \qquad (17)$$

Since ε can be made arbitrarily small, it follows that $\oint_{\triangle} f(z)dz = 0$.

4.2 Some Consequences of Cauchy Theorem

THEOREM **5.** *Let Γ_1 be a simple contour and Γ_2 be another simple closed contour lying entirely with Γ_1. If the single valued function $f(z)$ in a simply connected domain $\mathcal{D}$ be analytic in the region $\mathcal{R}$ bounded by the simple closed curves Γ and Γ_1, then*

$$\oint_{\Gamma_1} f(z)dz = \oint_{\Gamma_2} f(z)dz$$

where Γ_1 and Γ_2 are both traversed in the positive sense relative to their interior.

Proof: We join Γ_1 and Γ_2 by a straight line segment AB (Fig. 9). Obviously, the region $\mathcal{R}$ enclosed

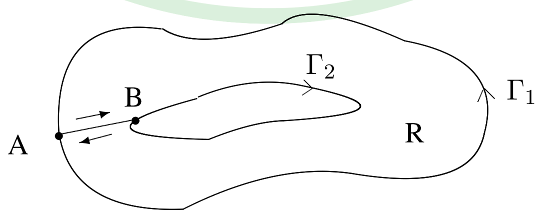

Figure 9: Simply connected

by the contour Γ is simply connected and the single valued function $f(z)$ is analytic in the region $\mathcal{R}$ and also on Γ so by Cauchy theorem, $\oint_{\Gamma} f(z)dz = 0$, i.e.,

$$\begin{aligned}&\int_{\Gamma_1} f(z)dz - \int_{\Gamma_2} f(z)dz + \int_{\overline{AB}} f(z)dz + \int_{\overline{BA}} f(z)dz = 0 \\ \text{or,}\quad &\int_{\Gamma_1} f(z)dz - \int_{\Gamma_2} f(z)dz + \int_{\overline{AB}} f(z)dz - \int_{\overline{AB}} f(z)dz = 0 \\ \text{or,}\quad &\int_{\Gamma_1} f(z)dz - \int_{\Gamma_2} f(z)dz = 0. \qquad (18)\end{aligned}$$

EXAMPLE **5.** *Evaluate* $\int_\Gamma \frac{dz}{z-a}$, *where* Γ *is a simple closed curve and* $z = a$ *is inside* Γ.

Solution: Here $1/(z-a)$ is analytic in a domain consisting of the complex plane excluding $z = a$. Let z is inside the contour Γ and Γ_1 be a circle of radius ε with centre at $z = a$. Now, on $\Gamma_1 : |z-a| = \varepsilon$, $z = a + \varepsilon e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Then, by the consequence (18), it follows that

$$\int_\Gamma \frac{dz}{z-a} = \int_{\Gamma_1} \frac{dz}{z-a} = \int_0^{2\pi} \frac{\varepsilon i e^{i\theta}}{\varepsilon e^{i\theta}} d\theta = 2\pi i,$$

which is the required value.

THEOREM **6.** *Let* $f(z)$ *be analytic in a simply-connected region* D *and let* z_0 *be a point in* $\mathcal{D}$. *Then the function, defined by* $F(z) = \int_{z_0}^{z} f(z^*)dz^*$ *for each* z *in* $\mathcal{D}$ *and* $F'(z) = f(z)$, *for each* z *in* $\mathcal{D}$.

Proof: $f(z)$ being analytic in $\mathcal{D}$, is continuous in $\mathcal{D}$, therefore, corresponding to arbitrary positive number ε, $\exists$ a positive number $\delta(\varepsilon)$, such that

$$|f(z^*) - f(z)| < \varepsilon; \text{ whenever } z^* \in \mathcal{D} \cap N(z, \delta). \tag{19}$$

We can choose as path the straight line segment joining z and $z + \Delta z$ provided we choose $|\Delta z|$ is small enough so that this path lies in $\mathcal{D}$. So

$$\Big|\int_z^{z+\Delta z} [f(z^*) - f(z)]du\Big| < \varepsilon|\Delta z|; \text{ whenever } |\Delta z| < \delta.$$

Therefore, we have,

$$\begin{aligned}\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) &= \frac{1}{\Delta z}\Big[\int_{z_0}^{z+\Delta z} f(z^*)du - \int_{z_0}^{z} f(z^*)du\Big] - f(z) \\ &= \frac{1}{\Delta z}\int_z^{z+\Delta z} [f(z^*) - f(z)]du.\end{aligned}$$

Therefore,

$$\begin{aligned}\Big|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\Big| &= \frac{1}{|\Delta z|}\Big|\int_z^{z+\Delta z} [f(z^*) - f(z)]du\Big| \\ &\leq \frac{1}{|\Delta z|}\int_z^{z+\Delta z} |[f(z^*) - f(z)]||du| \\ &\leq \frac{\varepsilon}{|\Delta z|}|\Delta z| = \varepsilon \text{ for } |\Delta z| < \delta.\end{aligned}$$

This, however, amounts to saying that

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z)$$

i.e., $F(z)$ is analytic and $F'(z) = f(z)$.

THEOREM **7** (Morera's theorem)**.** *Let* $f(z)$ *be continuous in a simply connected region* $\mathcal{D}$ *and let* $\oint_\Gamma f(z)dz = 0$, *where* Γ *any rectifiable closed Jordan curve in* $\mathcal{D}$. *Then* $f(z)$ *is analytic in* $\mathcal{D}$.

5 Cauchy's Integral Formula

THEOREM **8.** *Let f be analytic, regular within a simply connected region $\mathcal{D}$ and Γ is any simple closed contour lying entirely within $\mathcal{D}$. Then for any point z_0 interior to Γ,*

$$f(z_0) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z - z_0}\, dz, \tag{20}$$

where Γ is traversed in the positive (counter clockwise) sense.

Proof: Let $\mathcal{D}$ be a simply connected domain, Γ is a simple closed contour in $\mathcal{D}$, and z_0 an interior point of Γ. About the point $z = z_0$, let us describe a positively oriented small circle γ of radius r, defined by

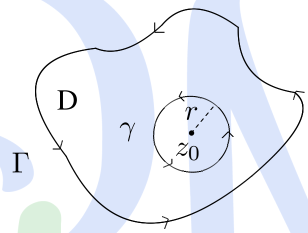

Figure 10: Cauchy's integral formula

the equation $\gamma : z - z_0 = re^{i\theta}$ where $\theta \in [0, 2\pi]$, lying entirely within Γ (Fig.10). Now, the function $\phi(z) = \dfrac{f(z)}{z - z_0}$ is analytic between and on the contours Γ and γ. Hence by the principle of deformation of contours, we get

$$\oint_\Gamma \phi(z)\ dz = \oint_\gamma \phi(z)\ dz. \tag{21}$$

$$\therefore \quad \oint_\Gamma \phi(z)dz = \oint_\Gamma \frac{f(z)}{z - z_0}dz = \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta$$
$$= i \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

Taking the limit $r \to 0$ of both sides, and making use of the continuity of $f(z)$, we have,

$$\begin{aligned}
\oint_\Gamma \phi(z)dz &= \lim_{r\to 0} i \int_0^{2\pi} f(z_0 + re^{i\theta})d\theta \\
&= i \int_0^{2\pi} \lim_{r\to 0} f(z_0 + re^{i\theta})d\theta = 2\pi i f(z_0) \\
\text{or, } f(z_0) &= \frac{1}{2\pi i} \oint_\Gamma \phi(z)\, dz = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z - z_0} dz.
\end{aligned}$$

This is known as *Cauchy's integral formula*. It tells us that if a function f is to be analytic with in and on a simple closed contour Γ, then the values of f interior to Γ are completely determined by the values of f on Γ. The formula (20) gives the value of $f(z)$ point each point z of $\mathcal{D}$ interior of an integral along Γ.

EXAMPLE **6.** *Using Cauchy's integral formula, evaluate the integral*

(i) $\displaystyle\int_\Gamma \frac{e^{2z}}{(z-1)(z-2)}\, dz$, *where Γ is the circle $|z| = 3$.*

(ii) $\displaystyle\int_{\Gamma} \frac{\cosh(\pi z)}{z(z^2+1)}\, dz$, *where* Γ *is the circle* $|z| = 2$.

Solution: (i) The function $f(z) = e^{2z}$ is analytic within the circle $\Gamma : |z| = 3$ and the singular points $z_0 = 1$ and $z_0 = 2$ lie inside Γ. Therefore,

$$\begin{aligned}\int_{\Gamma} \frac{e^{2z}}{(z-1)(z-2)}\, dz &= \int_{\Gamma} e^{2z}\Big[\frac{1}{z-2} - \frac{1}{z-1}\Big]\, dz \\ &= 2\pi i \times \Big[\frac{1}{2\pi i}\int_{\Gamma} \frac{e^{2z}}{z-2}\, dz - \frac{1}{2\pi i}\int_{\Gamma} \frac{e^{2z}}{z-1}\, dz\Big] \\ &= 2\pi i[f(2) - f(1)], \text{ using Eq. 20} \\ &= 2\pi i e^4 - 2\pi i e^2 = 2\pi i(e^4 - e^2).\end{aligned}$$

(ii) The function $f(z) = \cosh(\pi z) = \cos(i\pi z)$ is analytic within the circle $\Gamma : |z| = 2$ and the singular points $z_0 = 0$, $z_0 = i$ and $z_0 = -i$ lie inside Γ. Therefore,

$$\begin{aligned}\int_{\Gamma} \frac{\cosh(\pi z)}{z(z^2+1)}\, dz &= \int_{\Gamma} \Big[\frac{1}{z} + \frac{-1/2}{z-i} + \frac{-1/2}{z+i}\Big] \cosh(\pi z)\, dz \\ &= 2\pi i \times \Big[\frac{1}{2\pi i}\int_{\Gamma} \frac{\cos(i\pi z)}{z} - \frac{1}{2}\frac{1}{2\pi i}\int_{\Gamma} \frac{\cos(i\pi z)}{z-i} - \frac{1}{2}\frac{1}{2\pi i}\int_{\Gamma} \frac{\cos(i\pi z)}{z+i}\Big] \\ &= 2\pi i\Big[f(0) - \frac{1}{2}f(i) - \frac{1}{2}f(-i)\Big], \text{ using Eq. 20} \\ &= 2\pi i\Big[\cos 0 - \frac{1}{2}\cos(i^2\pi) - \frac{1}{2}\cos^2(-i^2\pi)\Big] \\ &= 2\pi i\Big[1 + \frac{1}{2} + \frac{1}{2}\Big] = 4\pi i.\end{aligned}$$

THEOREM **9** (Cauchy's integral formula for derivatives :)**.** *Let* $f(z)$ *be analytic within and on a closed contour* Γ *of a simply connected domain* $\mathcal{D}$ *and* z_0 *be any point in* $\mathcal{D}$, *then* $f^{(n)}(z)$ *is also analytic and*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_{\Gamma} \frac{f(z)}{(z-z_0)^{n+1}} dz \;:\; n = 1, 2, 3, \cdots \tag{22}$$

where $f^{(n)}(z_0)$ *is the* n^{th} *derivative of* $f(z)$ *at* $z = z_0$.

EXAMPLE **7.** *Evaluate* $\displaystyle\int_{\Gamma} \frac{z^3}{(2z+i)^3} dz$ *where* Γ *is the unit circle.*

Solution: Let $f(z) = z^3$, then $f'(z) = 3z^2$ and $f''(z) = 6z$. Also, the point $z = -i/2$ lies inside Γ. Hence using Eq. (22), we get

$$\begin{aligned}\int_{\Gamma} \frac{z^3}{(2z+i)^3} dz &= \frac{1}{8}\int_{\Gamma} \frac{z^3}{(z+i/2)^3} dz = \frac{1}{8}\,\frac{2\pi i}{2!}\,\frac{2!}{2\pi i}\int_{\gamma} \frac{f(z)dz}{(z+i/2)^3} \\ &= \frac{1}{8}\,\frac{2\pi i}{2!}\, f''(-i/2) = \frac{2\pi i}{16}\,(-3i) = \frac{3\pi}{8}.\end{aligned}$$

EXAMPLE **8.** *Evaluate* $\displaystyle\int_{\Gamma} \frac{e^z + z\sinh z}{(z-\pi i)^2} dz$ *where* Γ *is the circle* $|z| = 4$.

Solution: Let $f(z) = e^z + z\sinh z$, so $f'(z) = e^z + z\cosh z + \sinh z$. Also, the point $z = \pi i$ lies inside γ. Hence using Eq. (22), we get

$$\begin{aligned}\int_\Gamma \frac{e^z + z\sinh z}{(z-\pi i)^2}dz &= 2\pi i \times \frac{1!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-\pi i)^2}dz \\ &= 2\pi i f'(\pi i) = 2\pi i\Big[e^{\pi i} + \pi i\cosh(\pi i) + \sinh(\pi i)\Big] \\ &= 2\pi i(-1-\pi i) = -2\pi i(1+\pi i).\end{aligned}$$

THEOREM **10** (Cauchy's estimate)**.** *Let $f(z)$ be analytic function regular within a positively oriented circle $\Gamma = \{z : |z - z_0| = r\}$, then*

$$|f^{(n)}(z)| \le \frac{M \cdot n!}{r^n}; \quad n = 0, 1, 2, \cdots \tag{23}$$

where M is a constant such that $|f(z)| \le M$ on Γ, i.e., M is a maximum value of $|f(z)|$ on Γ.

Proof: Since Γ is closed and bounded, it is compact. Hence there exist some $M \in \mathbb{R}^+$ such that $|f(z)| \le M$ on Γ. By using Cauchy's integral formula (22),

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi}\oint \frac{|f(z)|}{|z-z_0|^{n+1}}|dz|.$$

Now the equation of the circle can be written as $z + z_0 + re^{i\theta} : 0 \le \theta < 2\pi$, then

$$\begin{aligned}|f^{(n)}(z_0)| &\le \frac{n!}{2\pi}\oint_\Gamma \frac{|f(z)|}{|z-z_0|^{n+1}}|dz| \\ &\le \frac{n!}{2\pi}\int_0^{2\pi}\frac{M\cdot r}{r^{n+1}}d\theta = \frac{M\cdot n!}{r^n}.\end{aligned}$$

This is the well known *Cauchy inequality*, that gives the maximum upper bound of $|f^{(n)}(z_0)|$.

THEOREM **11.** *[Liouville's theorem on integral functions] A bounded entire function must be a constant.*

Proof. By hypothesis, $f(z)$ is bounded on $\mathbb{C}$, so there exists $M \in \mathbb{R}^+$ such that $|f(z)| \le M, \forall z \in \mathbb{C}$. Let z_1 and z_2 be any two points in the z-plane. Take the contour Γ to be a large circle with center at z_1 and radius r such that z_2 is interior to Γ. Then, by Cauchy's integral formula (22),

$$\begin{aligned}f(z_2) - f(z_1) &= \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)dz}{z-z_2} - \frac{1}{2\pi i}\oint_\Gamma\frac{f(z)dz}{z-z_1} \\ &= \frac{z_1-z_2}{2\pi i}\oint_\Gamma \frac{f(z)dz}{(z-z_1)(z-z_2)}.\end{aligned} \tag{24}$$

Now, the equation of Γ may be written as $|z - z_1| = r$, so we have,

$$\begin{aligned}|z-z_2| &= |z-z_1+z_1-z_2| \ge |z-z_1| - |z_1-z_2| \\ &\ge r - |z_1-z_2| \ge r - \tfrac{r}{2} = \tfrac{r}{2}\end{aligned}$$

if we choose r so large that $|z_1 - z_2| < r/2$. Since $|f(z)| < M$, from (24) we get,

$$\begin{aligned} |f(z_2) - f(z_1)| &= \frac{|z_1 - z_2|}{2\pi}\left|\oint_\Gamma \frac{f(z)dz}{(z-z_1)(z-z_2)}\right| \\ &\leq \frac{|z_1 - z_2|}{2\pi}\oint_\Gamma \frac{|f(z)|}{|z-z_1|\cdot|z-z_2|}\,dz \\ &\leq \frac{|z_1 - z_2|}{2\pi}\oint_\Gamma \frac{M}{r\cdot\frac{r}{2}}dz = \frac{M|z_2 - z_1|}{\pi r^2}\cdot 2\pi r \\ &\leq \frac{2|z_2 - z_1|M}{r} \to 0 \text{ as } r \to \infty. \\ \Rightarrow f(z_1) - f(z_2) = 0, &\quad i.e.,\ f(z_1) = f(z_2). \end{aligned} \tag{25}$$

Since, (25) holds for all values of z_1 and z_2, therefore $f(z)$ is constant.

6 Modulus Theorem

THEOREM **12** (Maximum modulus theorem/principle)**.** *Let $f(z)$ be non-constant analytic function in $\mathcal{D}$ and continuous on $\bar{\mathcal{D}}$. Then the maximum value of $|f(z)|$ occurs on Γ and never in the interior of Γ.*

EXAMPLE **9.** *Find the maximum of $|f(z)|$ in $|z| \leq 1$ for the functions $f(z)$ given by (a) $z^2 - 3z + 2$ (b) $\cos 3z$.*

Solution: (a) Let $f(z) = z^2 - 3z + 2$. Being polynomial, it is analytic. To evaluate $M = \max_{|z|\leq 1}|f(z)|$, let us take $z = e^{i\theta}$, then

$$\begin{aligned} |f(z)| &= \left|e^{2i\theta} - 3e^{i\theta} + 2\right| = |e^{i\theta}|\left|e^{i\theta} - 3 + 2e^{-i\theta}\right| \\ &= \left|(\cos\theta + i\sin\theta) - 3 + 2(\cos\theta - i\sin\theta)\right|;\ \text{as } |e^{i\theta}| = 1 \\ &= \sqrt{9(\cos\theta - 1)^2 + \sin^2\theta} = \sqrt{2}\sqrt{4\cos^2\theta - 9\cos\theta + 5}. \end{aligned}$$

By maximum modulus theorem, the maximum value of $|f(z)|$ is attained on the boundary $\Gamma : |z| = 1$. Thus $M = \max_{|z|\leq 1}|z^2 - 3z + 2| = \sqrt{2}$, and the maximum value is attained at $\theta = \pi/2$, i.e., $z = \pm i$ on Γ.

(b) Let $z = x + iy$, then

$$\begin{aligned} f(z) &= \cos 3z = \cos 3(x + iy) = \cos 3x\cos(3iy) - i\sin 3x\sin(3iy) \\ &= \cos 3x\ \cosh 3y - i\sin 3x\ \sinh 3y \\ \Rightarrow |f(z)|^2 &= \cos^2 3x\ \cosh^2 3y + \sin^2 3x\ \sinh^2 3y \\ &= \cos^2 3x\ \cosh^2 3y + (1 - \cos^2 3x)\ \sinh^2 3y \\ &= \cos^2 3x + \sinh^2 3y. \end{aligned}$$

Therefore, $|f(z)| = \sqrt{\cos^2 3x + \sinh^2 3y}$. By maximum modulus principle, the maximum of $|f(z)|$ is attained on the boundary $\Gamma : |z| = 1$. Now, $|\cos 3x| \leq 1$ and the maximum value of $\cos 3x$ is 1 occurs at $x = 0$. Thus

$$g(y) = \sqrt{1 + \sinh^2 3y}; \qquad -1 \leq y \leq 1.$$

Since $\sinh 3y$ is an increasing function of y, the maximum value of $\sinh^2 3y$ occurs at $y = \pm 1$ and $g(y)$ has the maximum $\sqrt{1+\sinh^2 3}$ at the points $z = \pm i$. Thus

$$\max_{|z|\leq 1} |f(z) = \sqrt{1+\sinh^2 3y} = \cosh 3.$$

7 Taylor's Series

THEOREM **13.** *[Taylor's theorem] Let $f(z)$ be analytic inside a circle Γ with center at z_0, and radius r, then for every point z within Γ,*

$$\begin{aligned} f(z) &= f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \cdots + \frac{(z-z_0)^n}{n!}f^{(n)}(z_0) + \cdots \\ &= f(z_0) + \sum_{n=1}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0) \end{aligned} \tag{26}$$

In particular, when, $z_0 = 0$, equation (26) reduces to

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} f^{(n)}(0) \tag{27}$$

which is the well-known *Maclaurin's series.*

EXAMPLE **10.** *Find the Taylor's series to represent* $\dfrac{z^2-1}{(z+2)(z+3)}$ *in* $|z| < 2$.

Solution: Using partial fraction method, we get

$$\begin{aligned} f(z) &= \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - 8\frac{8}{z+3} \\ &= 1 + \frac{3}{2}\Big(1+\frac{z}{2}\Big)^{-1} - \frac{8}{3}\Big(1+\frac{z}{3}\Big)^{-1} \\ &= 1 + \frac{3}{2}\Big[1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \cdots\Big] - \frac{8}{3}\Big[1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\Big] \\ &= \Big(1 + \frac{3}{2} - \frac{8}{3}\Big) + \Big(-\frac{3}{2^2} + \frac{8}{3^2}\Big)z + \Big(\frac{3}{2\cdot 2^2} - \frac{8}{3\cdot 3^2}\Big)z^2 + \cdots \\ &= -\frac{1}{6} + \sum_{n=1}^{\infty}(-1)^{n+1}\Big(\frac{8}{3^{n+1}} - \frac{3}{2^{n+1}}\Big)z^n, \end{aligned}$$

and the expansion is valid in $|z| < 2$.

THEOREM **14** (Laurent's theorem)**.** *Let a function $f(z)$ be analytic in a ring shaped $\mathcal{D}$ bounded by two concentric circles Γ_1 and Γ_2 with center z_0 and radii r_1 and r_2 respectively $(r_1 > r_2)$. Then for all z in $\mathcal{D}$,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n} \tag{28}$$

where,

$$\left.\begin{aligned} a_n &= \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f(w)}{(w-z_0)^{n+1}}dw; \qquad n = 0,1,2,\cdots \\ b_n &= \frac{1}{2\pi i}\oint_{\Gamma_2} (w-z_0)^{n-1} f(w)dw; \quad n = 1,2,3,\cdots \end{aligned}\right\}. \tag{29}$$

EXAMPLE **11.** *Find the Laurent's series expansion of* $f(z) = z^2 e^{1/z}$ *about* $z = 0$.

Solution: Here $f(z) = z^2 e^{1/z}$ is analytic at all points $z \neq 0$, but not analytic at the isolated singular point $z = 0$ and hence can be expressed in the Maclaurin series. Now

$$\begin{aligned} f(z) &= z^2\left[1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right] \\ &= z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots . \end{aligned}$$

The analytic part of the series converges for $|z| < \infty$. The principal part is valid for $|z| > 0$. Thus the series converges for z except at $z = 0$. This representation is the required Laurent's series expansion valid for $0 < |z| < \infty$.

EXAMPLE **12.** *Expand* $f(z) = 1/(z-3)$ *in a Laurent's series valid for* $0 < |z| < 3$ *and* $|z| > 3$.

Solution: Here $f(z)$ is analytic at every point in the open disc $0 < |z| < 3$ and hence can be expressed in a Maclaurin series. Thus

$$\begin{aligned} \frac{1}{z-3} &= -\frac{1}{3}\frac{1}{1-(z/3)} = -\frac{1}{3}\left[1 - \frac{z}{3}\right]^{-1}; \quad \frac{|z|}{3} < 1 \\ &= -\frac{1}{3}\left[1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots\right] \\ &= -\frac{1}{3} - \frac{1}{4}z - \frac{1}{27}z^2 - \frac{1}{81}z^3 - \cdots \end{aligned}$$

Since there is no principal part so it is actually Maclaurin series. The domain $|z| > 3$ consists of all points exterior to the circle $|z| = 3$. Rewriting $f(z)$, we get

$$\begin{aligned} f(z) &= \frac{1}{z-3} = \frac{1}{z}\cdot\frac{1}{1-(3/z)} = \frac{1}{z}\left[1 - \frac{3}{z}\right]^{-1} \\ &= \frac{1}{z}\left[1 + \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \cdots\right] \\ &= \frac{1}{z}\left[1 + \frac{3}{z} + \frac{9}{z^2} + \frac{27}{z^3} + \cdots\right] = \frac{1}{z} + \frac{3}{z^2} + \frac{9}{z^3} + \cdots . \end{aligned}$$

EXAMPLE **13.** *Expand* $f(z) = \dfrac{1}{(z+1)(z+3)}$ *in a Laurent's series valid for the region (i)* $|z| < 1$ *(ii)* $1 < |z| < 3$ *(iii)* $|z| > 3$ *(iv)* $0 < |z+1| < 2$.

Solution: Resolving $f(z)$ its partial fractions, we get,

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}.$$

i) If $|z| < 1$, we have, $|\frac{z}{3}| < \frac{1}{3} < 1$. Therefore, the Laurent's series expansion is valid for $|z| < 1$.

Hence

$$\begin{aligned} f(z) &= \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1} \\ &= \frac{1}{2}[1-z+z^2-z^3+\cdots] - \frac{1}{6}\left[1-\frac{z}{3}+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\cdots\right] \\ &= \left(\frac{1}{2}-\frac{1}{6}\right) - \left(\frac{1}{2}-\frac{1}{18}\right)z + \left(\frac{1}{2}-\frac{1}{54}\right)z^2 - \left(\frac{1}{2}-\frac{1}{162}\right)z^3+\cdots \\ &= \frac{1}{3}-\frac{4}{9}z+\frac{13}{27}z^2-\frac{40}{81}z^3+\cdots \end{aligned}$$

Since there is no principal part, so, this is a Taylor's series.

ii) When $|z|>1$, we have, $\frac{1}{|z|}<1$, therefore,

$$\begin{aligned} \frac{1}{2(z+1)} &= \frac{1}{2z\left(1+\frac{1}{z}\right)} = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} \\ &= \frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right] = \frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\cdots \end{aligned}$$

For $|z|<3$, we have,

$$\frac{1}{2(z+3)} = \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1} = \frac{1}{6}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right]$$

Then the required Laurent's series expansion valid for both $|z|>1$ and $|z|<3$, i.e., $1<|z|<3$ is given by

$$\begin{aligned} f(z) &= \frac{1}{2(z+1)}-\frac{1}{2(z+3)} = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1} \\ &= \frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\cdots-\frac{1}{6}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right] \\ &= \cdots+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\frac{z^3}{162}-\cdots \end{aligned}$$

which contains both analytic and principal parts.

iii) Let us consider $|z|>3$, then $\frac{1}{|z|}<\frac{1}{3}<1$ and $\frac{3}{|z|}<1$. The required Laurent's series expansion becomes,

$$\begin{aligned} f(z) &= \frac{1}{2(z+1)}-\frac{1}{2(z+3)} = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1} \\ &= \frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\cdots-\frac{1}{2z}\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\cdots\right] \\ &= \left(\frac{3}{2}-\frac{1}{2}\right)\frac{1}{z^2}+\left(\frac{1}{2}-\frac{9}{2}\right)\frac{1}{z^3}+\left(-\frac{1}{2}+\frac{27}{2}\right)\frac{1}{z^4}+\cdots \\ &= \frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\cdots \end{aligned}$$

which contain only the principal parts ($b_1=0$) of Laurent's series expansion of $f(z)$.

iv) Let $z+1=u$, then $0<|z+1|<2$ implies $0<|u|<2$, $u\neq 0$. Therefore, $\left|\frac{u}{2}\right|<1$ and so,

$$\begin{aligned} f(z) &= f(z)=\frac{1}{(z+1)(z+3)}=\frac{1}{u(u+2)}=\frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1} \\ &= \frac{1}{2u}\left[1-\frac{u}{2}+\left(1+\frac{u}{2}\right)^2-\left(1+\frac{u}{2}\right)^3+\cdots\right] \\ &= \frac{1}{2u}-\frac{1}{4}+\frac{u}{8}-\frac{u^2}{16}+\cdots \\ &= \frac{1}{2(z+1)}-\frac{1}{4}+\frac{1}{8}(z+1)-\frac{1}{16}(z+1)^2+\cdots \end{aligned}$$

which contain only one principal part of the Laurent's series expansion of $f(z)$ about $z=-1$.

8 Zeros of an Analytic Function

Let a function f be analytic in some disk $|z-z_0|<\delta$. If $f(z_0)=0$ and there exists $r\in\mathbb{N}$ such that $f'(z_0)=f''(z_0)=\cdots=f^{(r-1)}(z_0)=0$ and $f^{(r)}(z_0)\neq 0$, then the analytic function f is said to have a *zero of order* r at z_0. If $r=1$, i.e., $f(z_0)=0$ and $f'(z_0)\neq 0$, then z_0 is known as a *simple zero* of f.

THEOREM **15.** *Let $f(z)$ be an analytic function at z_0. Then $f(z)$ has a zero of order r at z_0 if and only if there is a function g satisfying $f(z)=(z-z_0)^r g(z)$, where $g(z)$ is analytic at z_0 and $g(z_0)\neq 0$.*

Proof: Necessary condition: Suppose $f(z)$ is analytic at z_0 and f has a zero of order r at z_0. Then for $r\in\mathbb{N}$, $f'(z_0)=f''(z_0)=\cdots=f^{(r-1)}(z_0)=0$ hold in some neighbourhood $\mathcal{N}(z_0,\delta)$. Applying Taylor's theorem

$$\begin{aligned} f(z) &= \frac{(z-z_0)^r}{r!}f^{(r)}(z_0)+\frac{(z-z_0)^{r+1}}{(r+1)!}f^{(r+1)}(z_0)+\cdots \\ &= (z-z_0)^r\left[\frac{f^{(r)}(z_0)}{r!}+\frac{f^{(r+1)}(z_0)}{(r+1)!}(z-z_0)+\cdots\right] \\ &= (z-z_0)^r g(z),\;\; z\in\mathcal{N}(z_0,\delta) \end{aligned}$$

where, $g(z)=\dfrac{f^{(r)}(z_0)}{r!}+\dfrac{f^{(r+1)}(z_0)}{(r+1)!}(z-z_0)+\cdots$, $z\in\mathcal{N}(z_0,\delta)$ being a polynomial is analytic in $\mathcal{N}(z_0,\delta)$ and $g(z_0)=\dfrac{f^{(r)}(z_0)}{r!}\neq 0$.

Sufficient condition : Let $f(z)=(z-z_0)^r g(z)$, where $g(z)$ is analytic at z_0 and $g(z_0)\neq 0$. As $g(z)$ is analytic at z_0, therefore it has a Taylor series representation given by

$$g(z)=g(z_0)+\frac{g'(z_0)}{1!}(z-z_0)+\frac{g''(z_0)}{2!}(z-z_0)^2+\cdots$$

in some neighbourhood $\mathcal{N}(z_0,\delta)$ of z_0. Therefore,

$$\begin{aligned} f(z) &= (z-z_0)^r g(z)=(z-z_0)^r\left[g(z_0)+\frac{g'(z_0)}{1!}(z-z_0)+\frac{g''(z_0)}{2!}(z-z_0)^2+\cdots\right] \\ &= g(z_0)(z-z_0)^r+\frac{g'(z_0)}{1!}(z-z_0)^{r+1}+\frac{g''(z_0)}{2!}(z-z_0)^{r+2}+\cdots, \qquad (30) \end{aligned}$$

when $z \in \mathcal{N}(z_0, \delta)$. We see that $\frac{f^{(r)}(z_0)}{r!} + \frac{f^{(r+1)}(z_0)}{(r+1)!}(z - z_0) + \cdots$ is Taylor series representation for $f(z)$ and hence $f'(z_0) = f''(z_0) = \cdots = f^{(r-1)}(z_0) = 0$ and $f^{(r)}(z_0) = r!g(z_0) \neq 0$. Hence, f has a zero of order r at z_0. For example

(i) Consider $f(z) = \sin z$, we know that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = z\Big[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\Big] = z\, g(z),$$

where $g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$ is analytic and $g(0) = 1 \neq 0$. Therefore, $z = 0$ is a simple zero of $\sin z$. Similarly, for $f(z) = z^2 \sin z, z \sin z^2$, $z = 0$ is a zero of order 3.

(ii) Consider $f(z) = (z - 2i)^5 (z + 3)^2 e^z$, then $2i, -3$ are zeros of f of order 5 and 2, respectively.

THEOREM **16.** *Suppose $f(z)$ is analytic in a region $\mathcal{D}$ and is not identically zero in $\mathcal{D}$. Then the set of all zeros of $f(z)$ is isolated.*

Proof: Let $z_0 \in \mathcal{D}$ be a zero for $f(z)$. We shall prove that there exists a neighbourhood $\mathcal{N}(z_0, \delta)$ such that this neighbourhood does not contain any other zero for $f(z)$. Let z_0 be a zero of order r for $f(z)$, then

$$f(z) = (z - z_0)^r\, g(z),$$

where $g(z)$ is analytic at z_0, and $g(z_0) \neq 0$. We claim that $\mathcal{N}(z_0, \delta)$ does not contain any other zero of $f(z)$. Suppose $z_1 \neq z_0$ is another zero for $f(z)$ in $\mathcal{N}(z_0, \delta)$. Then $d(z_1, z_0) = |z_1 - z_0| < \delta$ and $f(z_1) = 0$.

$$\therefore (z_1 - z_0)^r g(z_1) = 0 \Rightarrow g(z_1) = 0,$$

as $z_1 \neq z_0$. Now, since g is analytic at z_0, g is continuous at z_0. Thus, we can find a $\delta > 0$ such that

$$\begin{aligned} & d(z_1, z_0) < \delta \Rightarrow d(g(z_1), g(z_0)) < \frac{|g(z_0)|}{2} \\ \Rightarrow \quad & |g(z_0)| < \frac{|g(z_0)|}{2}, \end{aligned}$$

which is a contradiction. Thus $\mathcal{N}(z_0, \delta)$ contains no other zero of $f(z)$ and hence the set of all zeros of $f(z)$ is isolated.

8.1 Isolated Singular Points

If z_0 is an isolated singular point of $f(z)$, then there exists a deleted neighbourhood of z_0 inside which $f(z)$ is analytic. Hence in this region, we can expand the function $f(z)$ as a Laurent's series. Isolated singular points are further classified into three types:

(i) removable singular point $\quad(ii)$ pole $\quad(iii)$ essential singular point

8.2 Removable Singularity

If the Laurent's series expansion (28) of an analytic function has no principal part, i.e., all the coefficients b_n are zero, so that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \;\; 0 < |z-z_0| < R, \tag{31}$$

then the isolated singular point z_0 is called *removable singular point* of $f(z)$. In this case $\lim_{z\to z_0} f(z)$ exists finitely. For example,

(i) for the function $f(z) = \frac{\sin z}{z}$; $z \neq 0$ has a removable singularity at $z = 0$ because in the Laurent's series expansion

$$\frac{\sin z}{z} = \frac{1}{z}\left\{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right\} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

there is no principle part (i.e., $b_n = 0$; $n = 1, 2, \cdots$) so, at $z = 0$, $f(z)$ has the removable singularity. When $f(0) = 1$ is assigned, f becomes entire.

(ii) for the function $f(z) = \frac{1-\cos z}{z^2}$; $z \neq 0$ has a removable singularity at $z = 0$ because in the Laurent's series expansion

$$\frac{1-\cos z}{z^2} = \frac{1}{z^2}\left[1 - \left\{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right\}\right] = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots$$

there is no principle part (i.e., $b_n = 0$; $n = 1, 2, \cdots$) so, at $z = 0$, $f(z)$ has the removable singularity. When $f(0) = 1/2$ is assigned, f becomes entire.

8.3 Poles

If $f(z)$ has the Laurent series expansion of the form (28) in which the principal part has only a finite number of nonzero terms given by

$$\sum_{n=1}^{m} \frac{b_n}{(z-z_0)^n} = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m} \tag{32}$$

where $b_m \neq 0$, then the singularity $z = z_0$ is called a *pole* of order m of the function $f(z)$. In particular, if $m = 1$, then the pole is said to be *simple pole* or pole of order 1; if $m = 2$, the pole is a *double pole* or a pole of order 2. For other finite values of m, the corresponding pole is called a *multiple pole* of order m. For example,

(i) $f(z) = \sin z/z^2$ has the Laurent series expansion about the singularity $z = 0$ is given by

$$\frac{\sin z}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots,$$

for $0 < |z| < \infty$. From this series, we see that $b_1 \neq 0$ and so $z = 0$ is a simple pole of the function $f(z)$.

(ii) $f(z) = \sinh z/z^4$ has the Laurent series expansion about the singularity $z = 0$ is given by

$$f(z) = \frac{\sinh z}{z^4} = \frac{1}{z^3} + \frac{1}{3!z} + \frac{z}{5!} + \frac{z^3}{7!} + \cdots, 0 < |z| < \infty.$$

Therefore, $z = 0$ is a pole of $f(z)$ of order 3.

(iii) $f(z) = 1/(z-1)^2(z-3)$ has the Laurent series expansion valid for $0 < |z-1| < 2$ is given by

$$f(z) = -\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)} - \frac{1}{8} - \frac{z-1}{16} - \cdots.$$

Since $b_2 = -1/2 \neq 0$, we conclude that $z = 1$ is a pole of order 2.

8.4 Essential Singularity

If the principal part of the Laurent's series expansion (28) of the function contains an infinite number of terms, i.e., $b_m \neq 0$ and are infinite in number, then the isolated singular point z_0 is called an essential singular point of $f(z)$. In this case $\lim_{z \to z_0} f(z)$ does not exist.

For example,

(i) the function $\sin \frac{1}{z-1}$ has essential singularity at $z = 1$ because

$$\sin \frac{1}{z-1} = \frac{1}{z-1} - \frac{1}{3!}\frac{1}{(z-1)^3} + \frac{1}{5!}\frac{1}{(z-1)^5} - \cdots$$

has infinite number of the terms in negative powers of $(z-1)$.

(ii) $f(z) = e^{3/z}$ has the Laurent series expansion valid for $0 < |z| < \infty$ is given by

$$e^{3/z} = 1 + \frac{3}{z} + \frac{3^2}{2!z^2} + \frac{3^3}{3!z^3} + \cdots.$$

It contains an infinite number of nonzero terms. This shows that $z = 0$ is an essential singularity of f.

EXAMPLE **14.** *Determine and classify all the singularities of the functions:*
a) $f(z) = \dfrac{1}{(2\sin z - 1)^2}$ *b)* $f(z) = \dfrac{\cos \pi z}{(z - z_0)^2}$.

Solution: (a) The poles of $f(z)$ are given by putting the denominator equal to zero, i.e., by

$$\begin{aligned}(2\sin z - 1)^2 = 0 \Rightarrow \sin z = \frac{1}{2} \Rightarrow z &= 2k\pi + \frac{\pi}{6} \\ &= (2k+1)\pi - \frac{\pi}{6} \quad ; \ k \in \mathbb{Z}.\end{aligned}$$

Obviously, $z = 2k\pi + \frac{\pi}{6}, (2k+1)\pi - \frac{\pi}{6}$ $k \in \mathbb{Z}$ are poles of order 2.

(b) Poles of $f(z)$ are given by putting the denominator equal to zero, i.e., by

$$\begin{aligned} & (\sec \pi z)(z - z_0)^2 = 0 \\ \Rightarrow \ & \pi z = n\pi \text{ and } z = z_0 \\ \Rightarrow \ & z = n \text{ and } z = z_0 \ ; \ n \in \mathbb{Z}\end{aligned}$$

Since $(z - z_0) = 0$ gives $z = z_0$ repeated twice, so $z = z_0$ is a double pole. Obviously, $z = \infty$ is the limit point of these poles $z = n : n \in \mathbb{Z}$, hence $z = \infty$ is a non-isolated essential singularity.

References

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