14

HYPERGEOMETRIC FUNCTION

14.1. Pochhammer symbol. Definition.

Let n be a positive integer. Then Pochhammer symbol is denoted and defined by

$$(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1) \qquad \ldots(1)$$

with

$$(\alpha)_0 = 1. \qquad \ldots(2)$$

Deductions. By definition, we have

I. $$(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1) = \frac{1.2.3\ldots(\alpha-1)\alpha(\alpha+1)\ldots(\alpha+n-1)}{1.2.3\ldots(\alpha-1)}$$

$$= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \text{ as } \Gamma(p) = (p-1)\Gamma(p-1)$$

Thus, $$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \qquad \ldots(3)$$

II. $$(\alpha)_{n+1} = \alpha(\alpha+1)(\alpha+2)\ldots[\alpha+(n+1)-1]$$

$$= \alpha[(\alpha+1)(\alpha+2)\ldots(\alpha+1+n-1)] = \alpha(\alpha+1)_n$$

Thus, $$(\alpha)_{n+1} = \alpha(\alpha+1)_n. \qquad \ldots(4)$$

III. $$(\alpha+n)(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)(\alpha+n)$$

$$= \alpha(\alpha+1)\ldots(\alpha+n-1)(\alpha+n+1-1) = (\alpha)_{n+1}$$

Thus, $$(\alpha+n)(\alpha)_n = (\alpha)_{n+1}. \qquad \ldots(5)$$

14.2. General hypergeometric function. Definition.

The general hypergeometric function is denoted and defined by

$${}_mF_n(\alpha_1, \alpha_2, \ldots \alpha_m; \beta_1, \beta_2 \ldots \beta_n; x) = \sum_{r=1}^{\infty} \frac{(\alpha_1)_r (\alpha_2)_r \ldots (\alpha_m)_r}{(\beta_1)_r (\beta_2)_r \ldots (\beta_n)_r} \cdot \frac{x^r}{r!}. \qquad \ldots(1)$$

The general hypergeometric function is also denoted by

$${}_mF_n\left[\begin{matrix} \alpha_1, \alpha_2, \ldots \alpha_m; \\ \beta_1, \beta_2 \ldots \beta_n; \end{matrix} x\right]. \qquad \ldots(2)$$

Remark. We shall consider only two special cases of (1) in the present Chapter. These are given in the following two articles for $m = n = 1$ and $m = 2$, $n = 1$ respectively.

14.3. Confluent hypergeometric (or Kummer) function. Definition.

Confluent hypergeometric function is denoted by ${}_2F_1(\alpha; \beta; x)$ or $F(\alpha; \beta; x)$ or $M(\alpha, \beta, x)$ and is defined by

$$F(\alpha; \beta; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\beta)_r} \cdot \frac{x^r}{r!}. \qquad \ldots(1)$$

Remark. Sometimes we use the following modified definition of the confluent hypergeometric function

$$F(\alpha;\beta;x) = 1+\frac{x}{1.\beta}x+\frac{\alpha(\alpha+1)}{1.2.\beta(\beta+1)}.x^2+\ldots \qquad \ldots(2)$$

14.4. Hypergeometric function. Definition. **[Meerut 88, 90]**

Hypergeometric function is denoted by ${}_2F_1(\alpha;\beta;\gamma;x)$ or simply $F(\alpha,\beta;\gamma;x)$ and is defined by

$$F(\alpha,\beta;\gamma;x) = \sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r}.\frac{x^r}{r!}. \qquad \ldots(1)$$

Remark 1. The series on the R.H.S. of (1) is

$$= 1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}.x^2+\ldots \qquad \ldots(2)$$

In particular, if $\alpha = 1$, $\beta = \gamma$, then the series (2) takes the form

$$1+x+x^2+x^3+\ldots,$$

which is a geometric series. Since (2) reduces to a geometric series as a particular case, (2) is called hypergeometric series.

Remark 2. Sometimes we use the following modified definition of the hypergeometric series.

$$F(\alpha,\beta;\gamma;x) = 1$$

$$+\frac{\alpha.\beta}{\gamma}\frac{x}{1!}+\frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!}+\ldots \qquad \ldots(3)$$

Remark 3. Hypergeometric function $F(\alpha,\beta;c;x)$ can be put in the following different forms:

$$F(\beta,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x) \qquad \ldots(4)$$

$$= (1-x)^{-a}F\left(a,c-b;c;\frac{x}{x-1}\right) \qquad \ldots(5)$$

$$= (1-x)^{-b}F\left(b,c-a;c;\frac{x}{x-1}\right). \qquad \ldots(6)$$

14.5. Gauss's hypergeometric equation or Gauss's equation or hypergeometric equation. Definition.

$$x(1-x)(d^2y/dx^2)+\{\gamma-(\alpha+\beta+1)x\}(dy/dx)-\alpha\beta\gamma = 0$$

is called hypergeometric equation.

14.6. Solution of the hypergeometric eqution.

Proof. Refer solved Ex. 7 of Art. 8.6 of chapter 8 for solution.

14.7. Symmetric property of hypergeometric function.

Hypergeometric function does not change if the parameters α and β are interchanged, keeping γ fixed. Thus, $F(\alpha,\beta;\gamma;x) = F(\beta,\alpha;\gamma;x)$.

Proof. We have, by definition

$$F(\alpha,\beta;\gamma;x) = \sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r}.\frac{x^r}{r!} \qquad \ldots(1)$$

and

$$F(\beta,\alpha;\gamma;x) = \sum_{r=0}^{\infty}\frac{(\beta)_r(\alpha)_r}{(\gamma)_r}.\frac{x^r}{r!}. \qquad \ldots(2)$$

From (1) and (2), we have $F(\alpha,\beta;\gamma;x) = (\beta,\alpha;\gamma;x)$.

14.8. Differentiation of hypergeometric functions.

Show that $\frac{d}{dx}F(\alpha,\beta;\gamma;x) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1;\gamma+1;x)$

and deduce that

(*i*) $\frac{d^n}{dx^n}F(\alpha,\beta;\gamma;x) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n,\beta+1;\gamma+1;x)$.

(*ii*) $\left[\frac{d^n}{dx^n}F(\alpha,\beta;\gamma;x)\right]_{x=0} = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}$.

Proof. By definition, we have $F(\alpha,\beta;\gamma;x) = \sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!}x^r$.

Differentiating both sides w.r.t. 'x', we have

$$\frac{d}{dx}F(\alpha,\beta;\gamma;x) = \sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!}.rx^{r-1} = \sum_{r=1}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r(r-1)!}x^{r-1}$$

($\because$ the term with $r = 0$ vanishes)

$$= \sum_{m=0}^{\infty}\frac{(\alpha)_{m+1}(\beta)_{m+1}}{(\gamma)_{m+1}m!}x^m$$

(taking m as the new variable of summation such that $r = m+1$ *i.e.* $m = r-1$ so that when $r = 1$, $m = 0$, and $r = \infty$, $m = \infty$)

$$= \sum_{m=0}^{\infty}\frac{\alpha(\alpha+1)_{m+1}.\beta(\beta+1)_m}{\gamma(\gamma+1)_m m!}x^m, \text{ by Art. 14.1}$$

$$= \frac{\alpha\beta}{\gamma}\sum_{m=0}^{\infty}\frac{(\alpha+1)_m(\beta+1)_m}{(\gamma+1)_m m!}x^m = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1;\gamma+1;x).$$

$$\therefore \quad \frac{d}{dx}F(\alpha,\beta;\gamma;x) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1;\gamma+1;x). \qquad \ldots(1)$$

Deduction. (*i*) For each positive integer, we must show that

$$\frac{d^n}{dx^n}F(\alpha,\beta;\gamma;x) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n,\beta+n;\gamma+n;x) \qquad \ldots(2)$$

Since $\alpha = (\alpha)_1$, $\beta = (\beta)_1$ and $\gamma = (\gamma)_1$, (1) shows that (2) is true for $n = 1$. We now assume that (2) is true for a particular value of n (say $n = m$) so that

$$\frac{d^m}{dx^m}F(\alpha,\beta;\gamma;x) = \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}F(\alpha+m,\beta+m;\gamma+m;x). \quad ...(3)$$

Differentiating both sides of (3) w.r.t. 'x' we get

$$\frac{d^{m+1}}{dx^{m+1}}F(\alpha,\beta;\gamma;x) = \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{d}{dx}F(\alpha+m,\beta+m;\gamma+m;x)$$

$$= \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{(\alpha+m)(\beta+m)}{(\gamma+m)}F(\alpha+m+1,\beta+m+1;\gamma+m+1;x)$$

[using (1) for $\alpha+m$, $\beta+m$, $\gamma+m$ in place of α, β, γ respectively]

$$\therefore \frac{d^{m+1}}{dx^{m+1}}F(\alpha,\beta;\gamma;x) = \frac{(\alpha)_{m+1}(\beta)_{m+1}}{(\gamma)_{m+1}}F(\alpha+m+1,\beta+m+1;\gamma+m+1;x) \quad ...(4)$$

where we have used relation (5) of Art. 14.1. (4) shows that (2) is true for $m+1$. Thus if (2) is true for $n=m$, then (2) is also true $n=m+1$. Hence by mathematical induction, (2) is true for each positive integer.

Deduction (*ii*). Putting $x=0$ in (2), we have

$$\left[\frac{d^n}{dx^n}F(\alpha,\beta;\gamma;x)\right]_{x=0} = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n,\beta+n;\gamma+n;0)$$

$$= \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\left[\sum_{r=0}^{\infty}\frac{(\alpha+n)_r(\beta+n)_r}{(\gamma+n)_r}\frac{x^r}{r!}\right]_{x=0} = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}.$$

14.9. Integral representation for the hypergeometric function

$$F(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}dt$$

or $$F(\alpha,\beta;\gamma;x) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}dt \text{ if } \gamma>\beta>0.$$

Proof. By definition, we have

$$F(\alpha,\beta;\gamma;x) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\cdot\frac{x^n}{n!} = \sum_{n=0}^{\infty}(\alpha)_n\frac{\Gamma(\beta+n)}{\Gamma(\beta)}\cdot\frac{\Gamma(\gamma)}{\Gamma(\gamma+n)}\cdot\frac{x^n}{n!}, \text{by Art. 14.1}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma(\beta)}\sum_{n=1}^{\infty}(\alpha)_n\frac{\Gamma(\beta+n)\Gamma(\gamma-\beta)}{\Gamma(\beta+n+\gamma-\beta)}\cdot\frac{x^n}{n!}$$

[Multiplying and dividing by $\Gamma(\gamma-\beta)$]

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\sum_{n=0}^{\infty}(\alpha)_n\left\{\int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}dt\right\}\frac{x^n}{n!}$$

[where $\gamma-\beta>0$, $\beta+n>0$ so that $\gamma>\beta>0$

Also, $\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(p,q) = \displaystyle\int_0^1 t^{p-1}(1-t)^{q-1}dt$]

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\left(\sum_{n=0}^{\infty}(\alpha)_n\frac{(xt)^n}{n!}\right)dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt \quad ...(1)$$

[$\because$ the general term in the expansion of $(1-xt)^{-\alpha}$

$$= \frac{(-\alpha)(-\alpha-1)...(-\alpha-n+1)}{n!}(-xt)^n$$

$$= (-1)^n\frac{\alpha(\alpha+1)...(\alpha+n-1)}{n!}\times(-1)^n x^n t^n$$

$$= (\alpha)_n\frac{x^n t^n}{n!}, \text{ by Art. 14.1}]$$

Since $$B(\beta,\gamma-\beta) = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\beta+\gamma-\beta)} = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)},$$

$$\therefore \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} = \frac{1}{B(\beta,\gamma-\beta)}. \quad ...(2)$$

Using (2), (1) may be re-written as

$$F(\alpha,\beta;\gamma;x) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt. \quad ...(3)$$

Thus (1) and (3) are the required results.

14.10. Gauss Theorem. **[Kanpur 91]**

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$$

Proof. From Art. 14.9, for $x=1$ we have

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-t)^{-\alpha}\,dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-\alpha-1}\,dt = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\,\frac{\Gamma(\beta)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\beta+\gamma-\beta-\alpha)}$$

$$\left(\because \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\right)$$

$$= \frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}.$$

14.11. Vandermonde's theorem.

$$F(-n,\beta;\gamma;1) = \frac{(\gamma-\beta)_n}{(\gamma)_n}.$$

Proof. From Art. 14.10, with $\alpha=-n$, we have

$$F(-n,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\beta+n)}{\Gamma(\gamma+n)\Gamma(\gamma-\beta)}$$

$$= \frac{\Gamma(\gamma)\Gamma(\gamma-\beta+n-1)(\gamma-\beta+n-2)\ldots(\gamma-\beta)\Gamma(\gamma-\beta)}{(\gamma+n-1)(\gamma+n-2)\ldots\gamma\Gamma(\gamma)\Gamma(\gamma-\beta)}$$

$$= \frac{(\gamma-\beta+n-1)(\gamma-\beta+n-2)\ldots(\gamma-\beta)}{(\gamma+n-1)(\gamma+n-2)\ldots\gamma} = \frac{(\gamma-\beta)_n}{(\gamma)_n}, \text{ by Art. 14.1}$$

14.12. Kummer's theorem. **[Agra 1990]**

$$F(\alpha, \beta; \beta-\alpha+1; -1) = \frac{\Gamma(\beta-\alpha+1)\Gamma(\beta/2+1)}{\Gamma(\beta+1)\Gamma(\beta/2-\alpha+1)}.$$

Proof. From Art. 14.9, with $x = -1$ and $\gamma = \beta - \alpha + 1$, we get

$F(\alpha, \beta; \beta-\alpha+1; -1)$

$$= \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\alpha)\Gamma(\beta-\alpha+1-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\beta-\alpha+1-\beta-1}(1-t)^{-\alpha}\, dt$$

$$= \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\Gamma(1-\alpha)}\int_0^1 t^{\beta-1}(1-t^2)^{-\alpha}\, dt$$

$$= \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\Gamma(1-\alpha)}\int_0^1 (u^{1/2})^{\beta-1}(1-u)^{-\alpha}\, (du/2\sqrt{u})$$

(putting $t^2 = u$ so that $dt = du/2\sqrt{u}$)

$$= \frac{\Gamma(\beta-\alpha+1)}{2\Gamma(\beta)\Gamma(1-\alpha)}\int_0^1 u^{(\beta/2)-1}(1-u)^{1-\alpha-1}\, du = \frac{\Gamma(\beta-\alpha+1)}{2\Gamma(\beta)\Gamma(1-\alpha)}.\frac{\Gamma(\beta/2)\Gamma(1-\alpha)}{\Gamma(\beta/2+1-\alpha)}$$

$$\left(\because \int_0^1 u^{p-1}(1-u)^{q-1} = B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\right)$$

$$= \frac{\Gamma(\beta-\alpha+1).(\beta/2).\Gamma(\beta/2)}{\Gamma(\beta/2+1-\alpha)\beta\,\Gamma(\beta)}, \text{ on multiplying } N^r \text{ and } D^r \text{ by } \beta$$

$$= \frac{\Gamma(\beta-\alpha+1)\,\Gamma(\beta/2+1)}{\Gamma(\beta/2+1-\alpha)\,\Gamma(\beta+1)}$$

14.13. More about the confluent hypergeometric function and solution of confluent hypergeometric equation

The hypergeometric differential equation is

$$(x^2-x)y'' + [(1+\alpha+\beta)x-\gamma]y' + \alpha\beta y = 0. \quad \ldots(1)$$

Replacing x by x/β in (1), we get

$$x\left(1-\frac{x}{\beta}\right)y'' + \left\{\gamma - \left(1+\frac{\alpha+1}{\beta}\right)x\right\}y' - \alpha y = 0 \quad \ldots(2)$$

Its solution is represented by the function $F(\alpha, \beta; \gamma; x/\beta)$

When $\beta \to \infty$, the equation (2) reduces to

$$xy'' + (\gamma - x)y' - \alpha y = 0 \quad \ldots(3)$$

whose solution is given by $\displaystyle\lim_{\beta\to\infty} F\left(\alpha, \beta; \gamma; \frac{x}{\beta}\right)$. ...(4)

The equation (3) is known as the *confluent hypergeometric differential equation* or *Kummer's equation.*

Now,
$$\lim_{\beta\to\infty}\frac{(\beta)_r}{\beta^r} = \lim_{\beta\to 0}\frac{\beta(\beta+1)(\beta+2)\ldots(\beta+r-1)}{\beta.\beta.\beta\ldots r \text{ times}}$$

$$= \lim_{\beta\to\infty}\left(1+\frac{1}{\beta}\right)\left(1+\frac{2}{\beta}\right)\ldots\left(1+\frac{r-1}{\beta}\right) = 1$$

Hence solution (4) may be written as

$$\lim_{\beta\to\infty} F\left(\alpha, \beta; \gamma; \frac{x}{\beta}\right) = \lim_{\beta\to\infty}\sum_{r=0}^{\infty}\frac{(\alpha)_r\,(\beta)_r}{r!\,(\gamma)_r}\left(\frac{x}{\beta}\right)^r$$

$$= \lim_{\beta\to\infty}\sum_{r=0}^{\infty}\frac{(\alpha)_r\,(\beta)_r}{r!\,(\gamma)_r\beta^r}x^r = \sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!\,(\gamma)_r}x^r = F(\alpha; \gamma; x). \quad \ldots(5)$$

The function $F(\alpha; \gamma; x)$ is called the *confluent hypergeometric function.*

Solution of this differential equation may also be obtained directly by the series integration method. Considering the equation (3), we find that $x = 0$ is a removable (non-essential) singularity and so the series representing the solution can be developed about the point $x = 0$.

Solution of the confluent hypergeometric differential equation when x = 0 and γ is not an integer.

Let us consider the solution of the hypergeometric differential equation (3) in the ascending powers of x as

$$y = x^k(a_0 + a_1x + a_2x^2 + \ldots) = \sum_{r=0}^{\infty} a_0 x^{k+r}, \; a_0 \neq 0. \quad \ldots(6)$$

$$\therefore \quad y' = \sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1} \text{ and } y'' = \sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}$$

Now putting the values of y, y' and y'' in (3) we get

$$x\sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2} + (\gamma-x)\sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1} - \alpha\sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

or
$$\sum_{r=0}^{\infty} a_r[(k+r)(k+r-1)+\gamma(k+r)]x^{k+r-1} - \sum_{r=0}^{\infty}[a_r(k+r)+\alpha]x^{k+r} = 0. \quad \ldots(7)$$

which is an identity and so coefficients of various powers of x must be zero.

Equating the coefficients of x^{k-1} (lowest powers of x) to zero, we get

$a_0[k(k-1)+\gamma k] = 0$ or $k(k-1)+\gamma k = 0$ as $a_0 \neq 0$.

Hence $k = 0$ and $k = 1-\gamma$ are the roots of the indical equation.

Now equation to zero the coefficients of x^{k+i}, we get

$$[(k+i+1)(k+i)+\gamma(k+i+1)]a_{i+1} - [(k+i+\alpha)]a_i = 0.$$

$$\therefore \quad a_{i+1} = \frac{(k+i)+\alpha}{(k+i+1)(k+i+\gamma)}a_i. \quad \ldots(8)$$

Case I. When k = 0. Then (8) gives

(*iii*) and (*iv*) Proceed like part (*ii*) above.

(*v*) We have, by definition

$$ {}_2F_1(\alpha, \beta ; \gamma ; x) = 1 + \frac{\alpha.\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}.\frac{x^2}{2!} + \ldots \quad \ldots(1) $$

Replacing α, β, γ, *x* by 1, 1, 2 and $-x$ respectively in (1), we get

$$ {}_2F_1(1, 1 ; 2 ; 1-x) = 1 + \frac{1.1}{2}.\frac{(-x)}{1!} + \frac{1.2.1.2}{2.3}.\frac{(-x)^2}{2!} + \frac{1.2.3.1.2.3}{2.3.4}.\frac{(-x)^3}{3!} + \ldots $$

Multiplying both sides of the above equation by *x*, we get

$$ x\,{}_2F_1(1, 1 ; 2 ; -x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \text{ ad. inf.} = \log(1+x). $$

(*vi*) and (*vii*) Proceed like part (*v*) above

(*viii*) We have, by definition

$$ F(\alpha, \beta ; \gamma ; x) = 1 + \frac{\alpha.\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \ldots \quad \ldots(1) $$

Replacing α, β, γ, *x* by 1/2, 1/2.3/2 and x^2 respectively in (1), we get

$$ F\left(\frac{1}{2}, \frac{1}{2} ; \frac{3}{2} ; x^2\right) = 1 + \frac{(1/2).(1/2)}{(3/2)}.\frac{x^2}{1!} + \frac{(1/2)(3/2).(1/2)(3/2)}{(3/2).(5/2)}.\frac{x^4}{2!} + \ldots $$

$$ \therefore \quad xF\left(\frac{1}{2}, \frac{1}{2} ; \frac{3}{2} ; x\right) = x + 1^2.\frac{x^3}{3!} + 1^2.3^2.\frac{x^5}{5!} + 1^2.3^3.5^2.\frac{x^7}{7!} + \ldots = \sin^{-1} x. $$

(*ix*) We have, by definition

$$ F(\alpha, \beta ; \gamma ; x) = 1 + \frac{\alpha.\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \ldots \quad \ldots(1) $$

Replacing α, β, γ and *x* by 1/2, 1, 3/2 and $-x^2$ respectively in (1), we have

$$ F\left(\frac{1}{2}, 1 ; \frac{3}{2} ; -x^2\right) = 1 + \frac{(1/2).1}{(3/2)}.\frac{(-x^2)}{1!} + \frac{(1/2)(3/2).1.2}{(3/2).(5/2)}.\frac{(-x^2)^2}{2!} + \ldots $$

$$ \therefore \quad F\left(\frac{1}{2}, 1 ; \frac{3}{2} ; -x^2\right) = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \ldots\infty $$

$$ xF\left(\frac{1}{2}, 1 ; \frac{3}{2} ; -x^2\right) = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots\infty = \tan^{-1} x. $$

Ex. 2. *Show that if* $|x| < 1$ *and* $|x/(1-x)| < 1$, *then*

$$ {}_2F_1(\alpha, \beta ; \gamma ; x) = (1-x)^{-\alpha}\,{}_2F_1\left(\alpha, \gamma-\beta ; \gamma ; \frac{x}{x-1}\right) $$

or

$$ F(\alpha, \beta ; \gamma ; x) = (1-x)^{-\alpha}\,F\left[\alpha, \gamma-\beta ; \gamma ; \frac{-x}{1-x}\right]. $$

Sol. By integral representation for the hypergeometric function (refer Art. 14.9), we have

$$ {}_2F_1(\alpha, \beta ; \gamma ; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt. \quad \ldots(1) $$

Putting $u = 1 - t$ so that $dt = -du$ and $t = 1 - u$, (1) gives

$$ {}_2F_1(\alpha, \beta ; \gamma ; x) $$

$$ = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_1^0 (1-u)^{\beta-1}u^{\alpha-\beta-1}(1-x+xu)^{-\alpha}\,(-du) $$

$$ = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 u^{\gamma-\beta-1}(1-u)^{\beta-1}(1-x)^{-\alpha}\left\{1 + \frac{xu}{1-x}\right\}^{-\alpha} du $$

$$ = \frac{(1-x)^{-\alpha}\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 u^{\gamma-\beta-1}(1-u)^{\beta-1}\left\{1 - \frac{x}{x-1}u\right\}^{-\alpha} du. \quad \ldots(2) $$

Replacing β and *x* by $\gamma - \beta$ and $x/(x-1)$ in (1), we get

$$ {}_2F_1\left(\alpha, \gamma-\beta ; \gamma ; \frac{x}{x-1}\right) $$

$$ = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma[\gamma-(\gamma-\beta)]}\int_0^1 t^{\gamma-\beta-1}(1-t)^{\gamma-(\gamma-\beta)-1}\left\{1 - \frac{xt}{x-1}\right\}^{-\alpha} dt $$

$$ = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma(\beta)}\int_0^1 u^{\gamma-\beta-1}(1-u)^{\beta-1}\left\{1 - \frac{x}{x-1}u\right\}^{-\alpha} du. \quad \ldots(3) $$

Using (3), (2) reduces to

$$ {}_2F_1(\alpha, \beta ; \gamma ; x) = (1-x)^{-\alpha}\,{}_2F_1\left(\alpha, \gamma-\beta ; \gamma ; \frac{x}{x-1}\right). $$

Ex. 3. *Prove :* $P_n(x) = {}_2F_1\left(-n, n+1 ; 1 ; \frac{1-x}{2}\right)$.

Sol. $P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n$, by Rodrigue's formula

$$ = \frac{(-1)^n}{n!\,2^n}\frac{d^n}{dx^n}(1-x^2)^n = \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left\{(1-x)^n.\frac{(1+x)^n}{2^n}\right\} $$

$$ = \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n.\left\{1 - \frac{1-x}{2}\right\}^n\right] $$

$$ = \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n.\left\{1 - n\left(\frac{1-x}{2}\right) + \frac{n(n-1)}{2!}\left(\frac{1-x}{2}\right)^2 - \ldots\right\}\right] $$

[By the binomial theorem]

$$ = \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n - \frac{n}{2}(1-x)^{n+1} + \frac{n(n-1)}{2!\,2^2}(1-x)^{n+2} + \ldots\right] $$

$$ = \frac{(-1)^n}{n!}\left[(-1)^n n! - (-1)^n\frac{n}{2}\frac{(n+1)!}{1!}(1-x) + (-1)^n.\frac{n(n-1)}{2!\,2^2}.\frac{(n+2)}{2!}(1-x)^2 + \ldots\right] $$

$$ \left[\because \frac{d^n}{dx^n}(a-bx)^m = (-1)^n b^n.\frac{m!}{(m-n)!}(a-bx)^{m-n}\right] $$

$$= \frac{(-1)^{2n}}{n!}\left[n! - \frac{n(n+1)}{2}n!(1-x) + \frac{n(n-1)}{2!\,2^2}(n+2)(n+1)n!(1-x)^2 + \ldots\right]$$

$$= 1 + \frac{(-n)(n+1)}{1.1!}\left(\frac{1-x}{2}\right) + \frac{(-n)(-n+1)(n+1)(n+2)}{2.1.2!}\left(\frac{1-x}{2}\right)^2 + \ldots$$

$$= {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right), \text{ by definition.}$$

Ex. 4. *Show that* $P_n(\cos\theta) = \cos^n\theta\ {}_2F_1\left(-\frac{n}{2}, -\frac{n-1}{2}; 1; -\tan^2\theta\right)$.

Sol. From Laplace's first integral for $P_n(x)$, (Refer Art. 9.6 in chapter 9),

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}[x \pm \sqrt{x^2-1}\cos\phi]^n\, d\phi. \qquad \ldots(1)$$

Let $x = \cos\theta$. Then, we have $\sqrt{x^2-1} = \sqrt{(\cos^2\theta - 1)} = i\sin\theta$. With these values and taking positive sign in (1), we get

$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^{\pi}(\cos\theta + i\sin\theta\cos\phi)^n\, d\phi.$$

$$\therefore P_n(\cos\theta) = \frac{\cos^n\theta}{\pi}\int_0^{1}(1 + i\tan\theta\cos\phi)^n\, d\phi$$

$$= \frac{\cos^n\theta}{\pi}\int_0^{\pi}\left\{1 + i\tan\theta\cos\phi + \frac{n(n-1)}{2!}i^2\tan^2\theta\cos^2\phi + \ldots\right\}d\phi$$

(by the binomial theorem)

$$= \frac{\cos^n\theta}{\pi}\left[\int_0^{\pi}d\phi + 0 + \frac{n(n-1)}{2}i^2\tan^2\theta\left(2\int_0^{\pi/2}\cos^2\phi\, d\phi\right) + 0 + \frac{n(n-1)(n-2)(n-3)}{3.2.1}i^4\tan^4\theta\left(2\int_0^{\pi/2}\cos^4\phi\, d\phi\right) + \ldots\right]$$

$$\left[\because \int_0^{2a} f(x)\,dx = \begin{cases} 0, \text{ if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, \text{ if } f(2a-x) = f(x)\end{cases}\right]$$

$$= \frac{\cos^n\theta}{\pi}\left[\pi - \frac{n(n-1)}{2}\tan^2\theta \times 2 \times \frac{1}{2}\times\frac{\pi}{2} + \frac{n(n-1)(n-2)(n-3)}{3.2.1}\tan^2\theta \times \frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} + \ldots\right]$$

$$= \cos^n\theta\left[1 - \frac{\left(-\frac{n}{2}\right)\left(-\frac{n-1}{2}\right)}{1}\frac{(-\tan^2\theta)}{1!} + \frac{\left(-\frac{n}{2}\right)\left(-\frac{n}{2}+1\right)\left(-\frac{n-1}{2}\right)\left(-\frac{n-1}{2}+1\right)}{2}\frac{(-\tan^2\theta)^2}{2!} + \ldots\right]$$

$$= \cos^n\theta\ {}_2F_1\left(-\frac{n}{2}, -\frac{n-1}{2}; 1; \tan^2\theta\right), \text{ by definition.}$$

EXERCISES

1. Show that ${}_1F_1(\beta;\gamma;x) = \lim_{\beta\to\infty} {}_2F_1(\alpha,\beta;\gamma;x/\beta)$.

2. Show that

$${}_2F_1(\alpha,\beta;\beta-\alpha+1;-1) = \frac{\Gamma(1+\beta-\alpha)\Gamma(1+\beta/2)}{\Gamma(1+\beta)\Gamma(1+\beta/2-\alpha)} \text{ and deduce that}$$

$${}_2F_1(\alpha, 1-\alpha;\gamma;1/2) = \frac{\Gamma(\gamma/2)\Gamma(\gamma/2+1/2)}{\Gamma(\alpha/2+\gamma/2)\Gamma(1/2-\alpha/2+\gamma/2)}.$$

[Meerut 1986]

3. Evaluate the integral $\int_0^{\infty} e^{-sx}\,{}_1F_1(\alpha;\beta;x)\,dx$. [**Ans.** $(1/s)_2F_1(\alpha, 1; \beta; s]$

[**Hint.** Use Art. 14.15]

4. Prove that $F(\alpha, \beta+1; \gamma+1; x) - F(\alpha,\beta;\gamma;x)$

$$= \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}xF(\alpha+1, \beta+1; \gamma+2; x).$$

[Meerut 1988]

5. The complete elliptic integral of the first kind is

$$K = \int_0^{\pi/2}\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}. \text{ Show that } K = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;k^2\right).$$

6. The complete elliptic integral of the second kind is

$$E = \int_0^{\pi/2}\sqrt{(1-k^2\sin^2\phi)}\,d\phi. \text{ Show that } E = \frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;k^2\right), |k| < 1.$$

7. Prove the following relations :

(*i*) $F(\alpha-1, \beta-1;\gamma;x) - F(\alpha,\beta-1;\gamma;x) = \frac{(1-\beta)x}{\gamma}F(\alpha,\beta;\gamma+1;x)$

(*ii*) $\alpha F(\alpha+1,\beta;\gamma;x) - (\gamma-1)F(\alpha,\beta;\gamma-1;x) = (\alpha+1-\gamma)F(\alpha,\beta;\gamma;x)$.

8. Prove that $F(\alpha, \beta; \gamma; 1/2) = 2^{\alpha} F(\alpha, \gamma-\beta; \gamma; -1)$.

9. Show that (*i*) $e^x - 1 = xF(1;2;x)$. (*ii*) $(1+x/\alpha)e^x = F(\alpha+1;\alpha;x)$.

10. The incomplete Gamma function is defined by the equation

$$\gamma(\alpha, x) = \int_0^{\pi} e^{-t}t^{\alpha-t}\,dt, \alpha > 0.$$

Prove that $\gamma(\alpha, x) = \alpha^{-1}x^{\alpha}F(\alpha;\alpha+1;-x)$.

11. Prove that following relations :

(*i*) $\beta F(\alpha;\beta;x) = \beta F(\alpha-1;\beta;x) + xF(\alpha;\beta+1;x)$.

(*ii*) $\alpha F(\alpha+1;\beta;x) - (\beta-1)F(\alpha;\beta-1;x) = (\alpha-\beta+1)F(\alpha;\beta;x)$.

12. Prove the following relations :

(*i*) $F(\alpha,\beta;\gamma;x) - F(\alpha,\beta;\gamma-1;x) = -\frac{\alpha\beta x}{\gamma(\gamma-1)}F(\alpha+1,\beta+1;\gamma+1;x)$

(*ii*) $F(\alpha+1;\gamma;x) - F(\alpha;\gamma;x) = (x/\gamma)F(\alpha+1,\gamma+1;x)$.