modification ... following section.

5. Wolfe's Modified Simplex Method :

Let the quadratic programming problem be

$$\text{Maximize } z = f(x) = \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk} x_j x_k$$

subject to the constraints $\sum_{j=1}^{n} a_{ij} x_j \le b_i$ and $i = 1, 2, \ldots m; j = 1, 2, \ldots, n.$

where $c_{jk} = c_{kj}$ for all j and k, $b_i \ge 0$ for all $i = 1, 2, \ldots, m.$

Also, assume that the quadratic form $\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk} x_j x_k$ be negative semi-definite and then $f(X)$ is concave.

Therefore the conditions given in section 4.1 become necessary and sufficient for an optimal solution to the above quadratic programming problem.

Then Wolfe's iterative procedure may be outlined as below:

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Step 1 : First, convert the inequality constraints into equations by introducing slack variables S_i^2 for $i = 1, 2, ..., m$ and S_{m+j}^2, $j = 1, 2, ..., n$ respectively.

Step 2 : Then, construct the Lagrangian function

$$L(X, S, \lambda) = f(X) - \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i + S_i^2 \right) - \sum_{j=1}^{n} \lambda_{m+j} \left(-x_j + S_{m+j}^2 \right)$$

Differentiate $L(X, S, \lambda)$ partially with respect to the components of X, S and λ. Equating the first order partial derivatives to zero, derive Kuhn-Tucker conditions from the resulting equations.

Step 3 : Introduce the non-negative artificial variable w_j,

$j = 1, 2, ..., n$ in the Kuhn-Tucker conditions

$$c_j + \sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \lambda_{m+j} + w_j = 0, j = 1, 2, ..., n;$$

and construct the objective function

$$z_w = w_1 + w_2 + ... + w_n.$$

Step 4 : Obtain an initial B.L.S. to the L.P.P.

Min $z_w = w_1 + w_2 + ... + w_n$

or, Max $z_w^* = -w_1 - w_2 - ... - w_n$

subject to $\sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} \lambda_j a_{ij} + \lambda_{m+j} + w_j = -c_j, j = 1, 2, ..., n$

$$\sum_{i=1}^{n} a_{ij} x_j + x_{n+i} = b_i, i = 1, 2, ..., m$$

$$w_j, \lambda_i, \lambda_{m+j}, x_j \geq 0, \; i = 1, 2, ..., m; j = 1, 2, ... n$$

and satisfying the complementary slackness condition

$$\sum_{j=1}^{n} \lambda_{m+j} x_j + \sum_{i=1}^{m} \lambda_i x_{n+i} = 0,$$

where, $x_{n+i} = S_i^2$, $i = 1, 2, ..., m$.

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Step 5 : Use two-phase simplex method to obtain an optimal solution of the LPP of Step 4.

This solution is an optimal solution to the given QPP also.

Important remarks on Wolfe's Method :

1. Let us designate as complementary variable each pair of x_j and λ_{m+j} and like wise each pair of S_i^2 and λ_i.

   In this method the basic solution at any iteration may correspond to one of the two cases:

   I. For each j and each i the basis contains only one complementary variable. Such a basic solution is called a standard basic solution and satisfies the complementary slackness condition.

   II. For each j and each i the basis contains a basic pair of complementary variables. Such a basic solution is called a non-standard basic solution and may not satisfy the complementary slackness condition. Whenever a non-standard basic occurs, the selection procedure for entering variable into the basis seeks to re-establish the complementary slackness condition.

2. If the quadratic programming problem is given in the minimization form, then convert it into maximization one by suitable modification in $f(X)$ and the '$\geq$' constraints.

3. The solution of the above system is obtained by using Phase-I of two phase simplex method. Since our aim is to obtain a feasible solution, the solution does not require the consideration of Phase-II. The only necessary thing is to maintain the conditions $\lambda_i S_i^2 = 0 = \lambda_{m+j} x_j$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ all the time. This implies that if λ_i is in the basic solution with positive value, then S_i^2 can not present in the basis with positive value. Similarly, λ_{m+j} and x_j can not be positive simultaneously.

4. It should be noted that Phase-I will end in the usual manner with the sum of all artificial variables equals to zero only if the feasible solution exists.

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6. Beale's Method :

Another approach to solve a quadratic programming problem has been suggested by Beale (1959). This is a technique for solving QPP without using the Kuhn–Tucker conditions. It involves the partitioning the variables into basis and non-basic ones. At each iteration, the objective function is expressed in terms of the non-basic variables only.

Let the QPP be given in the form

Min. $f(X) = CX + \frac{1}{2}X^T QX$, subject to the constraints $AX = b, X \geq 0$. Without loss of generality, every QPP with linear constraints can be written in this form.

The Beale's iterative procedure for solving such type of QPP can be outlined in the following steps.

Step-1. Convert the maximization problem into that of minimization, if it is in maximization form. Introduce slack and/or surplus variables to convert the inequalities into strict equations.

Step-2. Start with any initial solution as the basic solution and let it be denoted by

$$X_B = (x_{B_1}, x_{B_2}, \ldots, x_{B_m}).$$

In fact, we can choose arbitrarily any of the m variables as basic ones so that the remaining n–m variables become non-basic. Let the non-basic variables be denoted by

$$X_{NB} = (x_{NB_1}, x_{NB_2}, \ldots, x_{NB_m}).$$

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Step-3. Express the basic variables $x_{B_1}, x_{B_2}, \ldots, x_{B_m}$ in terms of the non-basic variables $x_{NB_1}, x_{NB_2}, \ldots, x_{NB_{n-m}}$ only, using the given and additional constraints, if any.

Step-4. Express $f(X)$ in terms of X_{NB} only, using the given and additional constraints, if any.

Step-5. Evaluate the partial derivatives of $f(X)$ with respect to the non-basic variables at $X_{NB} = 0$.

Step-6. See the nature of $\left(\frac{\partial f}{\partial x_{NB_k}}\right)_{X_{NB}=0}, k = 1, 2, \ldots, n-m.$

(i) If $\left(\frac{\partial f}{\partial x_{NB_k}}\right)_{X_{NB}=0} \geq 0$ for all $k = 1, 2, \ldots, n-m$, then the current solution is optimal.

(ii) If $\left(\frac{\partial f}{\partial x_{NB_k}}\right)_{X_{NB}=0} < 0$ for at least one k, then the current solution is not optimal. The non-basic variable corresponding to the maximum of $\left|\left(\frac{\partial f}{\partial x_{NB_k}}\right)_{X_{NB}=0}\right|$ will enter the basis.

Step-7. Let x_i be the entering variable identified in the above step. Determine how much x_i can be increased by calculating

(a) the largest value of x_i that can be attained without deriving the present basic variable (say x_ℓ) negative.

(b) that value of x_i at which $\frac{\partial f}{\partial x_{NB_i}}$ vanishes.

Step-8. Choose the minimum of the values obtained in (a) and (b) above. If it occurs for (a), x_ℓ will leave the basis. If it occurs for (b), no basic variable can be removed from the basis. In this case, we enlarge X_{NB} by introducing a new unrestricted variable u_i called a free variable defined by $u_i = \frac{1}{2}\frac{\partial f}{\partial x_i}$.

This will replace the non-basic variable x_i that has become a new basic variable. The equation $u_i = \frac{1}{2}\frac{\partial f}{\partial x_i}$ is to be treated as as additional constraint.

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Step-9. Got to step-3 and repeat the procedure until an optimal solution is reached. If X_{NB} contains a free variable u_i, then the optimality criteria includes an extra condition $\left(\frac{\partial f}{\partial u_i}\right)_{X_{NB}=0,u_i=0} = 0\ \forall i.$

Convergence : We give here a feasibility argument for the convergence of Beale's method in a finite number of steps. We shall say that the objective function is in standard form if the linear term contains no free variable.

For convergence we need the following procedure : If, at any stage, there is a non-basic free variable u for which $\frac{\partial f}{\partial y} = 0$, then a free variable must be removed from the non-basic set. Now, when f is in standard form its value in the present trial solution is a necessary value of f subject to the restriction that all the present non-basic restricted variables take the value zero, so there can be only one such value for any set of non-basic restricted variables. There are only a finite number of possible sets of non-basic restricted variables, so in order to show that the iterations, if not initially in standard form, must terminate, it suffices to show that they invariably reach a standard form in a finite number of steps.

We prove this as follows: our procedure ensures that when f is not in standard form a free variable will be removed from the non-basic set. Let p be the number of non-basic free variables. It is easy to show that, in the new expression for f, the off-diagonal elements in the row associated with the new non-basic variable must vanish.

It follows that f does not contain a linear term in this variable, and furtermore f can never contain a linear term in the variables unless some other restricted variable becomes non-basic, thereby decreasing p.

Therefore, if f is not in standard form, and $p = p_0$, then p can not increase and must decrease after at most p_0 steps, unless p meanwhile achieves a standard form. Since f is always in standard form when $p=0$, the required result follows.

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Example 6.1.2. Solve the following quadratic programming problem by Beale's method.

Maximize $Z = 2x_1 + 3x_2 - x_1^2$

subject to $x_1 + 2x_2 \leq 4$,

and $x_1, x_2 \geq 0$.

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Solution. At first, we convert the given maximization problem into the following minimization problem

Minimize $f(x_1, x_2) = -2x_1 - 3x_2 + x_1^2$

subject to $x_1 + 2x_2 \le 4, x_1, x_2 \ge 0.$

Introducing the slack variable x_3, the given constraint becomes

$$x_1 + 2x_2 + x_3 = 4,$$

$$x_1, x_2, x_3 \ge 0.$$

Now we arbitrarily choose x_1 as basic variable, so that

$X_B = (x_1)$ and $X_{NB} = (x_2, x_3)$

The basic variable x_1 can be expressed in terms of the non-basic variables x_2 and x_3 as

$$x_1 = 4 - 2x_2 - x_3$$

$$\therefore f(x_2, x_3) = -2(4 - 2x_2 - x_3) - 3x_2 + (4 - 2x_2 - x_3)^2$$

$$\therefore \frac{\partial f}{\partial x_2} = 4 - 3 + 2(4 - 2x_2 - x_3)(-2)$$

$$= 8x_2 + 4x_3 - 15$$

and $$\frac{\partial f}{\partial x_3} = +2 - 2(4 - 2x_2 - x_3)$$

$$= 4x_2 + 2x_3 - 6$$

$$\therefore \left(\frac{\partial f}{\partial x_2}\right)_{x_2=0, x_3=0} = -15$$

and $$\left(\frac{\partial f}{\partial x_3}\right)_{x_2=0, x_3=0} = -6$$

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∴ the current solution is not optimal. The maximum among the absolute values of the above two partial derivatives is 15 and it is corresponding to the non-basic variable x_2. So, x_2 will enter the basis.

In order to determine how much x_2 should or may be increased, we calculate the following two quantities:

a) the largest value of x_2 that can be attained without deriving the present basic variable negative.

Since $x_1 = 4 - 2x_2 - x_3$ and $x_3 = 0, x_1$ will become negative if x_2 is chosen greater than 2.

b) the value of x_2 for which $\frac{\partial f}{\partial x_2}$ vanishes.

The minimum of these two values of x_2 is $\frac{15}{8}$ which is obtained in case (b). So, x_1 can not be removed from the basis. In this case, we enlarge X_{NB} by introducing the free variable

$$u_2 = \frac{1}{2}\frac{\partial f}{\partial x_2} = -\frac{15}{2} + 4x_2 + 2x_3$$

or, $$x_2 = \frac{1}{4}\left(u_2 - 2x_3 + \frac{15}{2}\right)$$

Now we start the next iteration with $X_B = (x_1, x_2)$ and $X_N = (u_2, x_3)$.

Expressing x_1 in terms of u_2 and x_3, we have

$$x_1 = 4 - \frac{1}{2}\left(u_2 - 2x_3 + \frac{15}{2}\right) - x_3$$

$$= \frac{1}{4} - \frac{1}{2}u_2$$

Also, $$x_2 = \frac{1}{4}\left(u_2 - 2x_3 + \frac{15}{2}\right)$$

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$$\therefore f(u_2, x_3) = -2\left(\frac{1}{4} - \frac{1}{2}u_2\right) - 3\left(\frac{1}{4}u_2 - \frac{1}{2}x_3 + \frac{15}{3}\right) + \left(\frac{1}{4} - \frac{1}{2}u_2\right)^2$$

$$= \frac{1}{4}u_2^2 + \frac{3}{2}x_3 - \frac{97}{16}$$

$$\therefore \frac{\partial f}{\partial u_2} = \frac{1}{2}u_2$$

and $\frac{\partial f}{\partial x_3} = \frac{3}{2}$

$$\therefore \left(\frac{\partial f}{\partial u_2}\right)_{u_2=0, x_3=0} = 0$$

and $\left(\frac{\partial f}{\partial x_3}\right)_{u_2=0, x_3=0} = \frac{3}{2} > 0$.

Hence the optimality condition is reached and the optimal solution is $x_1^* = \frac{1}{4}, x_2^* = \frac{15}{8}, x_3^* = 0, u_2^* = 0$ and

$$z_{max} = -f_{min} = +2\left(\frac{1}{4}\right) + 3\left(\frac{15}{8}\right) - \left(\frac{1}{4}\right)^2 = \frac{97}{16}$$

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