12.6. CLASSIFICATION OF INVENTORY MODELS

The inventory problems (models) may be classified into two categories.

(i) Deterministic inventory models

These are the inventory models in which demand is assumed to be known constant or variables (dependent on time, stock-level, selling price of the item, etc.). Here, we shall consider deterministic inventory models for known constant demand. Such models are usually referred to as Economic lot-size models or Economic Order Quantity (EOQ) models.

(ii) Probabilistic inventory models

These are the inventory models in which the demand is a random variable having a known probability distribution. Here, the future demand is determined by collecting data from the past experience.

❑ 112.6.1. Deterministic inventory models

There are different types of models under this category, namely

(a) Purchasing inventory model with no shortage
(b) Manufacturing inventory model with no shortage
(c) Purchasing inventory model with shortages
(d) Manufacturing model with shortages
(e) Multi-item inventory model
(f) Price break inventory model

○ 6.1.1 Purchasing inventory model with no shortage (Model-1)

In this model, we want to derive the formula for the optimum order quantity per cycle of a single product so as to minimize the total average cost under the following assumptions and notations:

(i) Demand is deterministic and uniform at a rate D units of quantity per unit time.
(ii) Production is instantaneous (i.e., production rate is infinite).
(iii) Shortages are not allowed.
(iv) Lead time is zero.
(v) The inventory planning horizon is infinite and the inventory system involves only one item and one stocking point.
(vi) Only a single order will be placed at the beginning of each and the entire lot is delivered in one batch.
(vii) The inventory carrying cost, C_1 per unit quantity per unit time, the ordering cost, C_3 per order are known and constant.
(viii) T be the cycle length and Q, the ordering quantity per cycle.

Let us assume that an enterprise purchases an amount of Q units of item at time t = 0. This amount will be depleted to meet up the customer's demand. Ultimately, the stock level reaches to zero at time t = T. The inventory situation is shown in the Fig. – 1.

Clearly, $Q = DT$ (1)

Now, the inventory carrying cost for the entire cycle T is C_1 ×(area of ΔAOB) = $C_1 . (\frac{1}{2}QT)$ = $\frac{1}{2}C_1QT$ and the ordering cost for the said cycle T is C_3.

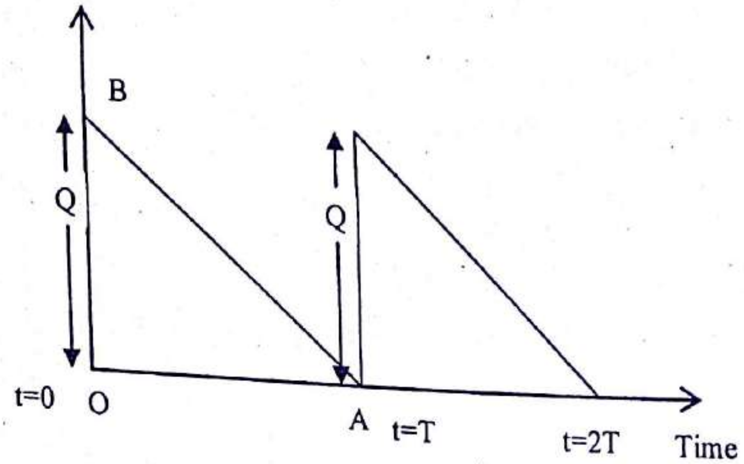

Fig. - 1

Hence the total cost for time T is given by

$$X = C_3 + \frac{1}{2}C_1QT$$

Therefore, the total average cost is given by $C(Q) = \frac{X}{T}$

or, $C(Q) = \frac{C_3}{T} + \frac{1}{2}QC_1$

or, $C(Q) = \frac{C_3D}{Q} + \frac{1}{2}C_1Q \qquad \left[\because Q = DT \ \therefore T = \frac{Q}{D}\right]$ (2)

The optimum value of Q which minimizes C(Q) is obtained by equating the first derivative of $C(Q)$ with respect to Q to zero.

i.e., $\frac{dC}{dQ}=0$ or, $\frac{1}{2}C_1-\frac{C_3D}{Q^2}=0$

or, $Q=\sqrt{\frac{2C_3D}{C_1}}$

Again, $\frac{d^2C(Q)}{dQ^2}=\frac{2C_3D}{Q}Q^2=\sqrt{\frac{2C_3D}{C_1}}$ which is + ve for $Q=\sqrt{\frac{2C_3D}{C_1}}$

Hence C(Q) is minimum for which the optimum value of Q is

$$Q^*=\sqrt{\frac{2C_3D}{C_1}} \qquad (3)$$

This is known as economic lot size formula or EOQ formula. The corresponding optimum time interval is $T^*=\frac{Q^*}{D}=\sqrt{\frac{2C_3}{C_1D}}$

and the minimum cost per unit time is given by $C_{\min}=\frac{C_3D}{Q^*}+\frac{1}{2}C_1Q^*=\sqrt{2C_1C_3D}$.

This model was first developed by Ford Harris of the Westing House Corporation, USA, in the year 1915. He derived the well-known classical lot size formula (3). This formula was also developed independently by R.H. Wilson after few years and it has been named as Harris – Wilson formula.

Remark :

(i) The total inventory time units for the entire cycle T is $\frac{1}{2}QT$, so the average inventory at any time is

$\frac{1}{2}QT/T=\frac{1}{2}Q$

(ii) Since $C_1>0$ from $f(Q)=\frac{1}{2}C_1Q$ it is obvious that the inventory carrying cost is a linear function of Q with a + ve slope i.e., for smaller average inventory, the inventory carrying costs are lower. In contrast, $g(Q)=\frac{C_3D}{Q}$ i.e., ordering cost increases as Q decreases.

(iii) In the above model, if we always maintain an inventory B on hand as buffer stock, then the average inventory at any time is $\frac{1}{2}Q+B$. Therefore, the total cost per unit time is

$$C(Q)=\left(\frac{1}{2}Q+B\right)C_1+C_3\frac{D}{Q}$$

As before, we obtain the optimal values of Q and T as follows:

$$Q=Q^*=\sqrt{\frac{2C_3D}{C_1}} \text{ and } T=T^*=\sqrt{\frac{2C_3}{DC_1}}$$

(iv) In the above model, if the ordering cost is taken as $C_3 + bQ$ (where b is the purchase cost per unit quantity) instead of fixed ordering cost then there is no change in the optimum order quantity.

Proof: In this case, the average cost is given by

$$C(Q)=\frac{1}{2}C_1Q+\frac{D}{Q}(C_3+bQ) \qquad (4)$$

The necessary condition for the optimum of $C(Q)$ in (4), we have

$$C'(Q)=0 \text{ implies } Q=\sqrt{\frac{2C_3D}{C_1}} \text{ and } C''(Q)>0.$$

Hence $Q^*=\sqrt{\frac{2C_3D}{C_1}}$

This shows that there is no change in Q^* in spite of change in the ordering cost.

Example 1 :

An engineering factory consumes 5000 units of a component per year. The ordering, receiving and handling costs are Rs. 300 per order while the trucking cost is Rs. 1200 per order, Interest cost Rs. 0.06 per unit per year, Deterioration and obsolence cost Rs. 0.004 per unit per year and storage cost Rs. 1000 per year for 5000 units. Calculate the economic order quantity and minimum average cost.

Solution : In the given problem, we have demand (D) = 5000 units

Ordering cost / Replenishment cost = Ordering, receiving, handling costs and trucing costs = Rs. (300 + 1200) = Rs. 1500 per order

Inventory carrying cost = interest costs + Deterioration and Obsolence costs + Storage costs

$$=\left(0.06+0.004+\frac{1000}{5000}\right)\text{rupees per unit per year} = \text{Rs. } 0.264 \text{ per unit per year}$$

Hence the economic order quantity is given by

$$Q^* = \sqrt{\frac{2C_3D}{C_1}} = \sqrt{\frac{2\times1500\times5000}{0.264}} = 753.8\,(approx.)$$

Also, the minimum average cost is

$$\sqrt{2C_1C_3D} = Rs.\sqrt{2\times0.264\times1500\times5000} = Rs.1989.97\,(Approx.)$$

6.1.2 Manufacturing model with no shortages or economic lot-size model with finite rate of replenishment and without shortage (Model – 2)

2013

In this model, we shall derive the formula for the optimum production quantity per cycle of a single product so as to minimize the total average cost under the following assumptions and notations:

(i) Demand is deterministic and uniform at a rate D unit of quantity per unit time.

(ii) Shortages are not allowed.

(iii) Lead time is zero.

(iv) The production rate or replenishment rate is finite, say, K units per unit time $(K > D)$.

(v) The production – inventory planning horizon is infinite and the production system involves only item and one stocking point.

(vi) The inventory carrying cost, C_1 per unit quantity per unit time, the setup cost, C_3 per production cycle are known and constant.

(vii) T be the cycle length and Q, the economic lot size.

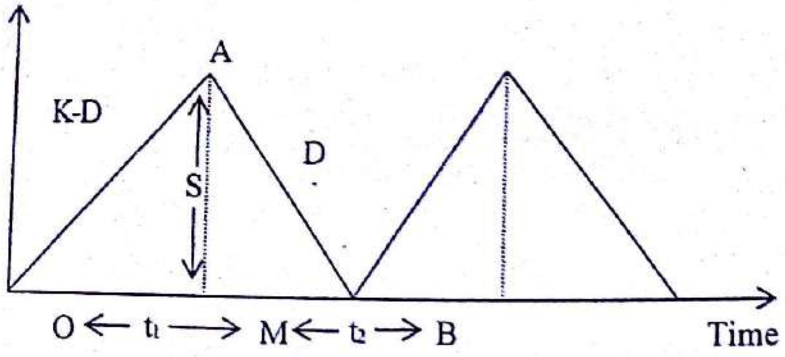

Fig. - 2

In this model, each production cycle time T consists of two parts t_1 and t_2 where

(i) t_1 is the period during which the stock is growing up a constant rate $K - D$ units per unit time.

(ii) t_2 is the period during which there is no replenishment (or production) but inventory is decreasing at the rate of D units per unit time.

Further, it is assumed that S is the stock available at the end of time t_1 which is expected to be consumed during the remaining period t_2 at the consumption rate D.

Therefore, $(K - D)t_1 = S$

or, $$t_1 = \frac{S}{K - D} \quad (5)$$

Since the total quantity produced during the production period t_1 is Q,

$$\therefore Q = Kt_1$$

or, $Q = K\dfrac{S}{K - D}$ which implies $$S = \frac{K - D}{K}Q \quad (6)$$

Again, $Q = DT$ i.e., $T = \dfrac{Q}{D}$

Now the inventory carrying cost for the entire cycle T is $(\Delta OAB)C_1 = \frac{1}{2}TSC_1$

and the setup cost for time period T is C_3.

Therefore, the total cost for the entire cycle T is given by $X = C_3 + \frac{1}{2}C_1ST$

Therefore the total average cost is given by $C(Q)=\frac{X}{T}$

Or, $C(Q)=\frac{C_3}{T}+\frac{1}{2}C_1S$

Or, $C(Q)=\frac{C_3D}{Q}+\frac{1}{2}C_1\frac{K-D}{K}Q \quad \left[\because Q=DT \text{ and } S=\frac{K-D}{K}Q\right]$ (7)

The optimum of Q which minimizes C(Q) is obtained by equating the first derivative of C(Q) with respect to Q to zero

i.e., $\frac{dC}{dQ}=0$

Or, $-\frac{C_3D}{Q^2}+\frac{1}{2}C_1\frac{K-D}{K}=0$

Or, $Q=\sqrt{\frac{2C_3}{C_1}\cdot\frac{DK}{K-D}}$ (8)

Again, $\frac{d^2C}{dQ^2}=\frac{2C_3D}{Q^3}$ = + ve quantity for $Q=\sqrt{\frac{2C_3}{C_1}\cdot\frac{DK}{K-D}}$

Hence C(Q) is minimum for which the optimum value of Q is

$$Q^*=\sqrt{\frac{2C_3}{C_1}\cdot\frac{DK}{K-D}} \quad (9)$$

The corresponding time interval is

$$T^*=\frac{Q^*}{D}=\sqrt{\frac{2C_3K}{C_1D(K-D)}} \quad (10)$$

and the minimum average cost is given by

$$C_{min}=\frac{1}{2}\frac{K-D}{K}C_1Q^*+\frac{C_3D}{Q^*}=\sqrt{2C_1C_3D\frac{K-D}{K}} \quad (11)$$

Remark:

(i) For this model, Q^*, T^* and C_{min} can be written in the following form :

$$Q^*=\sqrt{\frac{2C_3D}{C_1}\frac{1}{1-D/K}}, \quad T^*=\sqrt{\frac{2C_3}{DC_1}\frac{1}{1-D/K}} \quad (12)$$

If $K \to \infty$ i.e., the production rate is infinite, this model reduces to Model – 1. Therefore, when $K \to \infty$, then Q^*, T^* and C_{min} reduce to the expressions for Q^*, T^* and C_{min} of Model – 1.

❍ 6.1.3 Purchasing inventory model with shortages (Model – 3)

In this model, we shall derive the optimal order level and the minimum average cost under the following assumptions and notations:

(i) Demand is deterministic and uniform at a rate D unit of quantity per unit time.

(ii) Production is instantaneous (i.e., production rate is infinite).

(iii) Shortages are allowed and fully backlogged.

(iv) Lead time is zero.

(v) The inventory planning horizon is infinite and the inventory system involves only one item and one stocking point.

(vi) Only a single order will be placed at the beginning of each cycle and the entire lot is delivered in one batch.

(vii) The inventory carrying cost, C_1 per unit quantity per unit time, the shortage cost, C_2 per unit quantity per unit time, the ordering cost, C_3 per order are known and constant.

(viii) Q be the lot-size per cycle where as S_1 is the initial inventory level after fulfilling the backlogged quantity of previous cycle and $Q - S_1$ be the maximum shortage level.

(ix) To be the cycle length or scheduling period whereas t_1 be the no shortage period.

According to the assumptions of (viii) and (ix), we have $Q = DT$.

Regarding the cycle length or scheduling period of the inventory system, two cases may arise:

Case – 1 : Cycle length or scheduling period T is constant.

Case – 2 : Cycle length or scheduling period T is a variable.

Case – 1 : In this case, T is constant i.e., inventory is to be replenished after every time period T. As t_1 be the no shortage period, $S_1 = Dt_1$, or, $t_1 = S_1 / D$.

Now, the inventory carrying cost during the period 0 to t_1 is

C_1 (Area of $\Delta OAB) = \frac{1}{2} C_1 S_1 t_1 = \frac{1}{2} C_1 S_1^2/D$

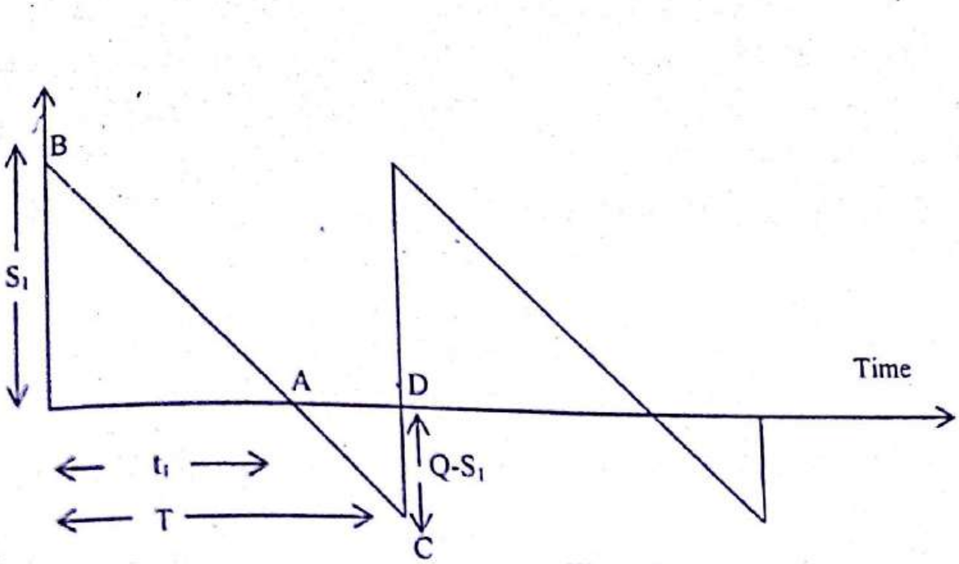

Fig. - 3

Again the shortage cost during the interval (t_1, T) is

$$C_2 \text{ (Area of } \Delta ACD) = \frac{1}{2} C_2 (Q - S_1)(T - t_1)$$

$$= \frac{1}{2} C_2 (Q - S_1)^2/D \quad \left[\because T - t_1 = \frac{Q - S_1}{D}\right]$$

Hence the total average cost of the system is given by

$$C = \left[\frac{1}{2} C_1 \frac{S_1^2}{D} + \frac{1}{2} C_2 \frac{(Q - S_1)^2}{D}\right] / T \tag{13}$$

Since the set-up cost C_3 and time period T are constant, the average set-up cost C_3/T also being constant will not be considered in the cost expression.

Since T is constant, Q = DT is also constant. Hence the above expression i.e., the expression for average cost is a function of single variable S_1. So, we can easily minimize the above expression (13) with respect to S_1 like Model - 1.

In this case, $S_1^* = \frac{C_2 Q}{C_1 + C_2} = \frac{C_2 DT}{C_1 + C_2}$ and $C_{min} = \frac{C_1 C_2 Q}{C_1 + C_2} = \frac{C_1 C_2 DT}{C_1 + C_2} \cdot \frac{1}{2}$ (14)

Case – 2 : In this case, cycle length or scheduling period T is a variable. Like Case - 1, the average cost of the inventory system will be

$$C=\left[C_3+\frac{1}{2}C_1\frac{S_1^2}{D}+\frac{1}{2}C_2\frac{(Q-S_1)^2}{D}\right]/T \qquad (15)$$

where $Q = DT$

Here, the average cost C is a function of two independent variables T and S_1.

Now, for optimal value of C, we have

$$\frac{\partial C}{\partial S_1}=0 \quad \text{and} \quad \frac{\partial C}{\partial T}=0$$

Now, $\frac{\partial C}{\partial S_1}=0$ gives $S_1 = C_2\frac{DT}{(C_1+C_2)}$ (16)

Again, $\frac{\partial C}{\partial T}=0$ gives $-\frac{C_1}{2D}\frac{S_1^2}{T^2}+C_2\frac{DT-S_1}{T}-\frac{C_2}{2D}\frac{(DT-S_1)^2}{T^2}-\frac{C_3}{T^2}=0$ (17)

Putting $S_1 = C_2\frac{DT}{(C_1+C_2)}$ in above and simplifying, we have

$$T=T^*=\sqrt{\frac{2C_3(C_1+C_2)}{C_1C_2D}} \qquad (18a)$$

Then,

$$S_1=S_1^*=\sqrt{\frac{2C_2C_3D}{C_1(C_1+C_2)}} \qquad (18b)$$

Obviously, for the values of T and S_1 given by (18a) and (18b),

$$\frac{\partial^2 C}{\partial S_1^2}>0,\ \frac{\partial^2 S}{\partial T^2}>0 \text{ and } \frac{\partial^2 C}{\partial S_1^2}\frac{\partial^2 C}{\partial T^2}-\left(\frac{\partial^2 C}{\partial S_1 \delta T}\right)^2>0$$

Hence C is minimum for the values of T and S_1 given by (18a) and (18b).

Therefore the optimum order quantity for minimum cost is given by

$$Q^*=DT^*=D\sqrt{\frac{2C_3(C_1+C_2)}{C_1C_2D}}=\sqrt{\frac{2C_3(C_1+C_2)D}{C_1C_2}} \qquad (19)$$

and

$$C_{\min}=C^*=\sqrt{\frac{2C_1C_2C_3D}{(C_1+C_2)}} \qquad (20)$$

Remark:

(i) If $C_1 \to \infty$ and $C_2 > 0$, inventories are prohibited. In this case $S_1^* = 0$ and each lot-size $Q^* = \sqrt{\frac{2C_3D}{C_2}}$ is used to fill the backorders.

(ii) If $C_2 \to \infty$ and $C_1 > 0$, then shortages are prohibited. In this case, $S_1^* = Q^* = \sqrt{\frac{2C_3D}{C_1}}$ and each batch Q^* is used entirely for inventory.

(iii) If shortage costs are negligible, then $C_1 > 0$ and $C_2 \to 0$.

In this case, $S_1^* \to 0$ and $Q^* \to \infty$.

(iv) If the inventory carrying costs are negligible, then $C_1 \to 0$ and $C_2 > 0$. In this case, $Q^* \to \infty$ and $S_1^* \to \infty$ i.e., $S_1^* \to Q^*$. Thus, due to very small inventory carrying costs, large lot size should be ordered and used to meet up the future demand.

(v) When the inventory carrying costs and shortage costs are equal i.e., when $C_1 = C_2$, $\frac{C_1}{C_1 + C_2} = \frac{1}{2}$.

In this case, $Q^* = \sqrt{2}\sqrt{\frac{2C_3D}{C_1}}$ which shows that the lot-size is √2 times of the lot-size of ***Model-1***.

Example-2 : 2013

The demand for an item is 18000 units per year. The inventory carrying cost is Rs. 1.20 per unit time and the cost of shortage is Rs. 5.00. The ordering cost is Rs. 400.00. Assuming that the replenishment rate is instantaneous, determine the optimum order quantity, shortage quantity, cycle length.

Solution : For the problem, it is given that demand (D) = 1800 units per year, carrying cost (C_1) = Rs. 1.20 per unit, shortage cost (C_2) = Rs. 5.00, ordering cost (C_3) = Rs. 400 per order.

The optimum order quantity Q^* is given by

$$Q^* = \sqrt{\frac{2C_3(C_1 + C_2)D}{C_1C_2}} = \sqrt{\frac{2 \times 400 \times (1.2 + 5) \times 18000}{1.2 \times 5}} = 3857 \text{units}$$

Again, the optimum shortage quantity $Q^* - S_1^* = 3857 - \sqrt{\dfrac{2C_2C_3D}{C_1(C_1+C_2)}}$

$$= 3857 - \sqrt{\frac{2\times5\times400\times18000}{1.2\times(1.2+5)}} = 746 \text{ units (Approx.)}$$

Optimal cycle length $T^* = \dfrac{Q^*}{D} = \dfrac{3857}{18000} = 0.214$ year (Approx.)

Example – 3 :

The demand for an item is deterministic and constant over time and it is equal to 600 units per year. The unit cost of the item is Rs. 50.00 while the cost of placing an order is Rs. 100.00. The inventory carrying cost is 20% of the unit cost of the item and the shortage cost per month is Re. 1. Find the optimal ordering quantity. If shortages are not allowed, what would be the loss of the company?

Solution: It is given that D = 600 units/ year

C_1 = 20% of Rs. 50.00 = Rs. 10.00

C_2 = Re 1.00 per month i.e., Rs. 12.00 per year

C_3 = Rs. 100.00 per order

When shortages are allowed, the optimal ordering quantity Q^* is given by

$$Q^* = \sqrt{\frac{2C_3(C_1+C_2)D}{C_1C_2}} = 148 \text{ units}$$

and the minimum cost per year is $C(Q^*) = \sqrt{2C_1C_2C_3D/(C_1+C_2)}$ = Rs. 809.04

If shortages are not allowed, then the optimal order quantity is

$$Q^* = \sqrt{\frac{2C_3D}{C_1}} = 109.5 \text{ units}$$

and the relevant average cost is given by $C(Q^*) = \text{Rs.}\sqrt{2C_1C_3D}$ = Rs. 1095.44.

Therefore, if shortages are not allowed, the loss of the company will be Rs. (109.44 - 809.04) i.e., Rs. 286.40.

6.1.4 Manufacturing model with shortage or Economic lot-size model with finite rate of replenishment and shortages (Model-4)

In this model, we shall derive the formula for the optimum production quantity, shortage quantity and cycle length of a single product by minimizing the average cost of the production system under the following assumptions and notations:

(i) The production rate or replenishment rate is finite, say K units per unit time ($K > D$).

(ii) The production – inventory planning horizon is infinite and the production system involves only one item and one stocking point.

(iii) Demand of the item is deterministic and uniform at a rate D unit of quantity per unit time.

(iv) Shortages are allowed.

(v) Lead time is zero.

(vi) The inventory carrying cost, C_1 per unit quantity per unit time, the shortage cost, C_2 per unit quantity per unit time and the set up cost, C_3 per set up are known and constant.

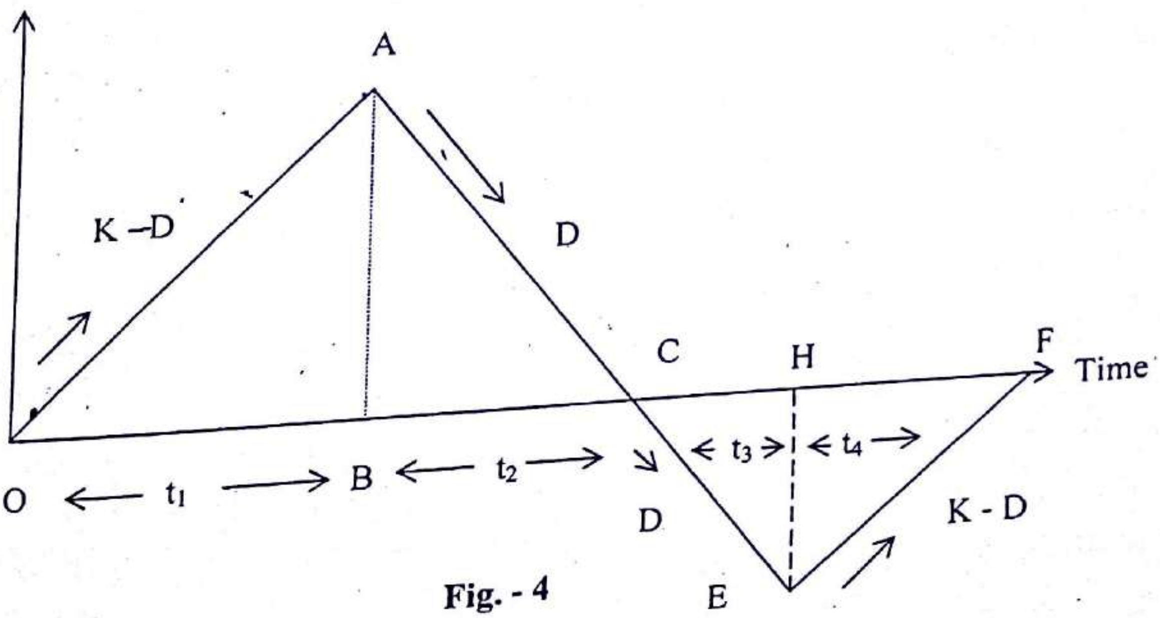

Fig. - 4

(vii) T be the cycle length of the system i.e., T be the interval between production cycle.

(viii) Q be the economic lot-size.

Let us assume that each production cycle of length T consists of two parts t_{12} and t_{34} which are further subdivided into t_1 and t_2, t_3 and t_4 where (i) inventory is building up at a constant rate K – D units per unit time during the interval $[0, t_1]$, (ii) at time $t = t_1$, the production is stopped and the stock level decreases due to meet up the customer's demand only upto the time $t = t_1 + t_2$, (iii) Shortages are accumulated at a constant rate of D units per unit time during the time t_3 i.e., during the interval $[t_{12}, t_{12} + t_3]$. (iv) Shortages are being filled up immediately at a constant rate K – D units per unit time during the time t_4 i.e., during the interval $[t_{12} + t_3, t_{34}]$. (v) The production cycle then repeats itself after the time $T = t_1 + t_2 + t_3 + t_4$.

Again, let the inventory level is S_1 at $t = t_1$ and at the end of time $t = t_1 + t_2$, the stock level reaches to zero. Now shortages start and suppose that shortages are build up of quantity S_2 at time $t = t_1 + t_2 + t_3$ and then these shortages be filled up upto the time $t = t_1 + t_2 + t_3 + t_4$. The pictorial representation of the inventory situation is given in Fig. – 4.

Now our objectives are to find the optimal value of Q, S_1, S_2, t_1, t_2, t_3, t_4 and T with the minimum average total cost.

Now the inventory carrying cost over the time period T is given by

$$C_h = C_1 \times \Delta OAC = C_1 \cdot \frac{1}{2} OC.AB = \frac{1}{2} C_1 (t_1 + t_2) S_1$$

and the shortage cost over time T is given by

$$C_s = C_2 \times \Delta CEF = C_2 \cdot \frac{1}{2} CF.EH = \frac{1}{2} C_2 (t_3 + t_4) S_2 .$$

Hence the total average cost of the production system is given by

$$C = [C_3 + C_h + C_s]/T$$

From Fig. – 4, it is clear that $S_1 = (K - D) t_1$ or, $t_1 = \frac{S_1}{K - D}$ (21)

Again, $S_2 = D t_2$ or, $t_2 = \frac{S_2}{D}$ (22)

Now, in stock-out situation, $S_2 = D t_3$ or, $t_3 = \frac{S_2}{D}$ (23)

and $S_2 = (K-D)t_4$ or, $t_4 = \frac{S_2}{K-D}$ (24)

Since the total quantity produced over the time period T is Q,

$Q = DT$ where D is the demand rate

or, $D(t_1 + t_2 + t_3 + t_4) = Q$

$$\text{or, } D\left(\frac{S_1}{K-D} + \frac{S_1}{D} + \frac{S_2}{D} + \frac{S_2}{K-D}\right) = Q \quad (25)$$

After simplification, we have $S_1 + S_2 = \frac{K-D}{K}Q$ (26)

Again, $t_1 + t_2 = \frac{K}{D(K-D)}S_1$ and $t_3 + t_4 = \frac{K}{D(K-D)}S_2$ (27)

Now substituting the values of $t_1 + t_2$, $t_3 + t_4$ and $T = Q/D$ in (21), we have

$$C(Q, S_1, S_2) = \frac{1}{2Q}\frac{K}{K-D}\left(C_1 S_1 + C_2 S_2^2\right) + \frac{DC_3}{Q} \quad (28)$$

Using (26), the above reduces to

$$C(Q, S_2) = \frac{1}{2Q}\frac{K}{K-D}\left[C_1\left(\frac{K-D}{K}Q - S_2\right)^2 + C_2 S_2^2\right] + \frac{DC_3}{Q} \quad (29)$$

Now, for the extreme values of $C(Q, S_2)$, we have

$$\frac{\partial C}{\partial Q} = 0, \quad \frac{\partial C}{\partial S_2} = 0$$

$$\frac{\partial C}{\partial Q} = 0 \text{ implies } S_2 = C_1 \frac{K-D}{K}\frac{Q}{C_1 + C_2} \quad (30)$$

$$\text{Again, } \frac{\partial C}{\partial S_2} = 0 \text{ gives } Q = \sqrt{\frac{2C_3(C_1 + C_2)}{C_1 C_2}}.\sqrt{\frac{KD}{K-D}} \quad (31)$$

For these values of Q and S_2 given in (31) and (30), it can easily be verified that

$$\frac{\partial^2 C}{\partial Q^2} > 0, \quad \frac{\partial^2 C}{\partial S_2^2} > 0 \text{ and } \frac{\partial^2 C}{\partial Q^2}\frac{\partial^2 C}{\partial S_2^2} - \left(\frac{\partial^2 C}{\partial Q \partial S_2}\right)^2 > 0$$

Hence $C(Q, S_2)$ is minimum and the optimal values of Q and S_2 are given by

$$Q^* = \sqrt{\frac{2C_3(C_1+C_2)}{C_1C_2}}\sqrt{\frac{KD}{K-D}} \quad (32)$$

and $$S_2^* = \sqrt{\frac{2C_1C_3}{C_2(C_1+C_2)}}\sqrt{\frac{D(K-D)}{K}} \quad (33)$$

$$T^* = \frac{Q^*}{D} = \sqrt{\frac{2C_3(C_1+C_2)}{C_1C_2}}\sqrt{\frac{K}{D(K-D)}} \quad (34)$$

$$S_1^* = \frac{K-D}{K}Q^* - S_2^* = \sqrt{\frac{2C_2C_3}{C_1(C_1+C_2)}}\sqrt{\frac{D(K-D)}{K}} \quad (35)$$

Now $$C_{\min} = C(Q^*, S_2^*) = \sqrt{\frac{2C_1C_2C_3}{C_1+C_2}}\sqrt{\frac{D(K-D)}{K}} \quad (36)$$

Remarks:

(i) In this model, if we assume that the production rate is infinite i.e., $K \to \infty$, then the optimal quantities by taking $K \to \infty$ in (32), (34) and (36) are

$$Q^* = \sqrt{\frac{2C_3(C_1+C_2)}{C_1C_2}}D, \quad T^* = \sqrt{\frac{2C_3(C_1+C_2)}{C_1C_2D}} \text{ and } C_{\min} = \sqrt{\frac{2C_1C_2C_3D}{C_1+C_2}}$$

This means that Model – 4 reduces to Model – 3 if $K \to \infty$.

(ii) If shortages are not allowed in Model – 4, then it reduces to Model – 3. In this case, taking $C_2 \to \infty$ in (32), (34) and (36) we obtain the required expressions of model – 3 which are as follows :

$$Q^* = \sqrt{\frac{2C_3KD}{C_1(K-D)}}, \quad T^* = \sqrt{\frac{2C_3K}{C_1D(K-D)}} \text{ and } C_{\min} = \sqrt{\frac{2C_1C_3D(K-D)}{K}}$$

Example – 4.

The demand for an item in a company is 18000 units per year. The company can produce the item at a rate of 3000 per month. The cost of one set-up is Rs. 500 and the holding cost of one unit per month is Rs. 0.15. The shortage cost of one unit is Rs. 20 per month. Determine the optimum manufacturing quantity and the shortage quantity. Also determine the manufacturing time and the time between setups.

Solution:

For this problem, it is given that

C_1 = Rs. 0.15 per month, C_2 = Rs. 20 per month, C_3 = Rs. 500.00 per setup, K = 3000 per month, D = 18000 units per year i.e., 1500 units per month

The optimum manufacturing quantity Q^* is given by

$$Q^* = \sqrt{\frac{2C_3(C_1+C_2)}{C_1C_2}}\sqrt{\frac{KD}{K-D}} = \sqrt{\frac{2\times 500\times(0.15+20)}{0.15\times 20}}\sqrt{\frac{3000\times 1500}{3000-1500}} = 4489 \text{ units(approx.)}$$

The optimum shortage quantity is given by

$$S_2^* = C_1\frac{K-D}{K}\frac{Q^*}{(C_1+C_2)} = 17 \text{ units (approx.)}$$

Manufacturing time = $\frac{Q^*}{K} = \frac{4489}{3000} = 1.5$ months and the time between setups $\frac{Q^*}{D} = \frac{4489}{1500} = 3$ months.