Linear equation of 2nd order:

The general form of the linear equⁿ of 2nd order is,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X, \quad \text{——— (1)}$$

where P, Q and X are constants or funⁿ of x.

Theorem: (A) Known integral method
If an integral included in the complementary funⁿ of a linear equⁿ of 2nd order be known, then the complete soluⁿ can be expressed in terms of that known integral.

Method to solve:

Let $y = u(x)$, be a known integral in the complementary function of (1).
Now we will find the complete soluⁿ of (1) in terms of $u(x)$.

Let us assume that $y = u(x)v(x)$ be the complete soluⁿ of (1). ——— (2)

$\therefore y = uv$

$\Rightarrow y_1 = u_1 v + u v_1$

and $y_2 = u_2 v + 2u_1 v_1 + u v_2$

suffixes denote the order of diff. w.r.t. x

Substituting the values of y, y_1, y_2 in (1),

$$u v_2 + (2u_1 + Pu)v_1 + (u_2 + Pu_1 + Qu)v = X \quad \text{——— (3)}$$

Since $y = u$ is a soluⁿ of (1) with $X = 0$ i.e.

$$y_2 + Py_1 + Qy = 0 \quad \text{——— (4)}$$

So, $y = u$ will satisfy (4) i.e. $u_2 + Pu_1 + Qu = 0$ ——— (5)

From (3) & (5) we get,

$$u v_2 + (2u_1 + Pu) v_1 = X$$

$$\Rightarrow v_2 + \left(2\frac{u_1}{u} + P\right) v_1 = \frac{X}{u} \quad \text{—— (6)}$$

which is a linear equⁿ & put $\frac{dv}{dx} = q$

$\therefore$ (6) $\Rightarrow$

$$\frac{dq}{dx} + \left(\frac{2}{u}\frac{du}{dx} + P\right) q = \frac{X}{u} \quad \text{—— (7)}$$

which is linear in q & x and 1st order.

$$I.F. = e^{\int [P + (2/u)\frac{du}{dx}] dx}$$

$$= e^{\int P dx} \cdot e^{2\log u} = u^2 \cdot e^{\int P dx}$$

and solⁿ of (7),

$$q u^2 e^{\int P dx} = \int \frac{\cancel{X}}{u} u^2 e^{\int P dx} \cdot dx + C_1$$

$$\Rightarrow q = \frac{dv}{dx} = \frac{e^{-\int P dx}}{u^2} \int \cancel{X} u e^{\int P dx} dx + \frac{C_1 e^{-\int P dx}}{u^2}$$

Integrating,

$$v = \int\left\{\frac{1}{u^2} e^{-\int P dx} \cdot \int R u e^{\int P dx} \cdot dx\right\} dx + C_1 \int \frac{1}{u^2} e^{-\int P dx} \cdot dx + C_2 \quad \text{—— (8)}$$

Putting value of v in (2),

$$\boxed{y = C_2 u + C_1 u \int \frac{1}{u^2} e^{-\int P dx} \cdot dx + u \int\left\{\frac{1}{u^2} e^{-\int P dx} \cdot \int R u e^{\int P dx} \cdot dx\right\} dx}$$

which is the required solⁿ.

~~[illegible]~~ Rule for finding an integral of (C.F.)

Write the given equⁿ in the standard form $y'' + Py' + Qy = X$, in which the coefficient of y'' is unity.

(a) Let $y = e^{mx}$ be a solⁿ of $y'' + Py' + Qy = 0$

$\Rightarrow (m^2 + Pm + Q)e^{mx} = 0$

$\Rightarrow m^2 + Pm + Q = 0$, since $e^{mx} \neq 0$

Therefore,

(i) $u = e^{ax}$ will be a solⁿ $(m=a)$ if $m^2 + Pm + Q = 0$

(ii) $u = e^{x}$ will be a solⁿ $(m=1)$ if $1 + P + Q = 0$

(iii) $u = e^{-x}$ will be a solⁿ $(m=-1)$ if $1 - P + Q = 0$

(b) Let $y = x^m$ be a solⁿ of $y'' + Py' + Qy = 0$

$\Rightarrow m(m-1)x^{m-2} + mPx^{m-1} + Qx^m = 0$

$\Rightarrow m(m-1) + mPx + Qx^2 = 0$ since $x^{m-2} \neq 0$.

Therefore,

(i) $u = x$ is a solⁿ $(m=1)$ if $P + Qx = 0$

(ii) $u = x^2$ is a solⁿ $(m=2)$ if $2 + 2Px + Qx^2 = 0$

(iii) $u = x^m$ is a solⁿ ~~(m=a)~~ if $m(m-1) + mPx + Qx^2 = 0$

Example :

① Solve $xy'' - (2x-1)y' + (x-1)y = 0$

Re-writting the given eqn. in standard form,

$$y'' - \left(2 - \frac{1}{x}\right)y' + \left(1 - \frac{1}{x}\right)y = 0 \quad \text{——— ①}$$

comparing ① with $y'' + Py' + Qy = 0$, we get,

$$P = -\left(2 - \frac{1}{x}\right) \text{ and } Q = 1 - \frac{1}{x} \quad \text{——— ②}$$

Here, $1 + P + Q = 1 - 2 + \frac{1}{x} + 1 - \frac{1}{x} = 0$.

$\therefore$ $u = e^x$ is a part of C.F. of the soln. of ①.
——— ③

Let the complete soln. of ① be, $y = v \cdot e^x$ ——— ④

Then v is given by $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{X}{u}$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(-2 + \frac{1}{x} + \frac{2}{e^x} \cdot e^x\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} = 0 \quad \text{——— ⑤}$$

Put $\frac{dv}{dx} = q$

⑤ $\Rightarrow$ $\frac{dq}{dx} + \frac{q}{x} = 0 \quad \Rightarrow \frac{dq}{q} = -\frac{dx}{x}$

$\Rightarrow \log q = -\log x + \log c_1 \quad \Rightarrow q = c_1/x$

$\Rightarrow \frac{dv}{dx} = c_1/x \Rightarrow dv = c_1 \frac{dx}{x}$

$\therefore$ $v = c_1 \log x + c_2$ ——— ⑥

$\therefore$ complete soln. of ① is $y = e^x(c_1 \log x + c_2)$.

(B) ~~[illegible]~~ Reducing into normal form

Method to solve: It is not always possible to find an integral soln. of C.F.. So in this method we will solve 2nd order linear equn. with no such known integral.

The general equn. is,

$$y'' + Py' + Qy = X \quad \text{--- (1)}$$

Let $y = u(x)\,v(x)$ is soln. of (1) & none of u & v is a part of C.F. of (1).

$\therefore\ y' = uv' + u'v$ & $y'' = uv'' + 2u'v' + u''v$

putting in (1),

$$uv'' + 2u'v' + vu'' + P(uv' + vu') + Quv = X$$

$$\Rightarrow uv'' + u\left(P + \frac{2}{u}u'\right)v' + (u'' + Pu' + Qu)v = X \quad \text{--- (2)}$$

To remove v', choose u in such a way, that $P + \frac{2}{u}u' = 0 \Rightarrow P + \frac{2}{u}\frac{du}{dx} = 0$ --- (3)

$$\Rightarrow \underline{u = e^{-\frac{1}{2}\int P\,dx}} \quad \text{--- (4)}$$

$$(2) \Rightarrow uv'' + (u'' + Pu' + Qu)v = X \quad \text{--- (5)}$$

$$(3) \Rightarrow u' = -\frac{Pu}{2} \ \& \ u'' = -\frac{Pu'}{2} - \frac{uP'}{2} = \frac{P^2u}{4} - \frac{u}{2}P'$$

putting u', u'' in (5),

$$v'' + \left(Q - \frac{1}{4}P^2 - \frac{1}{2}P'\right)v = Xe^{\frac{1}{2}\int P\,dx} \quad \text{--- (6)}$$

Solve (6) for v & put in $y = v \cdot e^{-\frac{1}{2}\int P\,dx}$

② Solve $x^2y'' - (x^2+2x)y' + (x+2)y = x^3e^x$

Re-writing in the standard form,

$$y'' - \left(1+\frac{2}{x}\right)y' + \left(\frac{1}{x} + \frac{2}{x^2}\right)y = xe^x \quad \text{—— ①}$$

Comparing ① with $y'' + Py' + Qy = X$, we have

$$P = -(1+2/x), \quad Q = \frac{1}{x} + \frac{2}{x^2} \ \& \ X = xe^x \quad \text{—— ②}$$

Here $P + Qx = 0$ showing that $u = x$ —— ③ is a part of C.F. of ①.

Let the general soln. of ① be, $y = uv$ —— ④

where v is, $\dfrac{d^2v}{dx^2} + \left(P + \dfrac{2}{u}\dfrac{du}{dx}\right)\dfrac{dv}{dx} = \dfrac{X}{u}$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(-1 - \frac{2}{x} + \frac{2}{x}.1\right)\frac{dv}{dx} = \frac{xe^x}{x}$$

$$\Rightarrow \frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x \quad \text{—— ⑤}$$

Let $\dfrac{dv}{dx} = q$

$\therefore$ ⑤ $\Rightarrow$ $\dfrac{dq}{dx} - q = e^x$ which is linear in q & x.

I.F. $= e^{\int -dx} = e^{x}$ and the soln. is,

$$qe^{-x} = \int e^x.e^{-x}.dx + c_1 = x + c_1$$

$$\Rightarrow q = (x+c_1)e^{-x} \Rightarrow dv = (x+c_1)e^{-x}.dx$$

$$\Rightarrow v = (x+c_1)e^{x} - \int 1.e^x dx + c_2$$

$$\Rightarrow v = (x+c_1)e^{x} - e^x + c_2 = (x+c_1-1)e^x + c_2$$

$\therefore$ Complete soln. $y = x\left[(x+c_1-1)e^x + c_2\right]$.

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Wronskian : Its Property & application :

Linearly dep. funⁿs : $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be linearly dep. on an interval T, if $\exists$ constants $c_1, c_2, \ldots, c_n$, not all zero, such that,

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0, \ \forall x \in T$$

e.g. $\{1, \sin^2 x, \cos^2 x\}$, $\{\sin x, \sin 3x, \sin^3 x\}$.

Linearly indep. funⁿs : $f_1(x), f_2(x), \ldots, f_n(x)$ are linearly indep. on T iff the constants which satisfy,

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \forall x \in T$$

are the constants $c_1 = c_2 = \cdots = c_n = 0$.

e.g. $\{e^x, e^{2x}, e^{3x}\}$, $\{e^x, \sin x, \cos x\}$, $\{1, \sin x, \cos x\}$

If $f_1, f_2, \ldots, f_n$ are linearly dep. then $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$ where all of c_n are not zero. Let $c_r \neq 0$ then,

$$f_r = \alpha_1 f_1 + \cdots + \alpha_{r-1} f_{r-1} + \alpha_{r+1} f_{r+1} + \cdots + \alpha_n f_n.$$

where $\alpha_i = -\dfrac{c_i}{c_r}$

So, f_r can be written as the linear combination of other $(n-1)$ funⁿs.

Wronskian : The wronskian of two differentiable funⁿs $f_1(x)$ & $f_2(x)$ on an interval T is defined by the determinant, $W(f_1(x), f_2(x)) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$

(where dash = derivative)

A wronskian with $f_1, f_2, \ldots, f_n$ is,

$$W(f_1(x), \ldots, f_n(x)) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{n-1} \end{vmatrix}$$

NOTE: for the fun$\underline{\underline{c}}$s $1, x, x^2, \ldots, x^{n-1}$; $n>1$

$W = 0!\, 1!\, 2! \cdots (n-1)!$

Non zero wronskian is a sufficient condition for n fun$\underline{\underline{c}}$s to be linearly indep.

Let the fun$\underline{\underline{c}}$s be $f_1, f_2, \ldots f_n$ & assume that they are atleast $(n-1)$ times differentiable.
Now for constants $c_1, c_2, \ldots c_n$ let,

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

After successive differentiation,

$$c_1 f_1' + c_2 f_2' + \cdots + c_n f_n' = 0$$

$$c_1 f_1^{n-1} + c_2 f_2^{n-1} + \cdots + c_n f_n^{n-1} = 0$$

$$\Rightarrow \begin{bmatrix} f_1 & \cdots & f_n \\ \vdots & & \\ f_1^{n-1} & \cdots & f_n^{n-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow AX = B$

if $|A| = 0 \Rightarrow X$ has non zero soln.
$\Rightarrow f_i$ are dep.
if $|A| \neq 0 \Rightarrow X$ has only zero soln.
$\Rightarrow f_i$ are indep.
$\therefore A \neq 0$ i.e. $W \neq 0$ for being f_i linearly indep.

Method of Variation of Parameters:

This method is used to find the complete primitive of a linear diff. equn (of any order), when its complementary func is known. The complete soln is obtained by varying the parameters of the C.F.

Let us consider the general linear equn of 2nd order,

$$y'' + Py' + Qy = X \quad \text{——— } (1)$$

where P, Q, X are constants or func. of x.

Let $y = Au + Bv$ ——— (2) be the C.F. of (1), where A & B are constants and u & v are func. of x and u & v are linearly indep. solutions of the corresponding homogeneous equn. Thus,

$$u'' + Pu' + Qu = 0 \;\; \& \;\; v'' + Pv' + Qv = 0 \quad \text{——— } (3)$$

Let us assume that $y = \phi u + \psi v$ ——— (4) is the complete primitive of (1), where ϕ, ψ in place of A & B and they are func. of x to be so chosen that (4) will satisfy (1).

Differentiating (4) w.r.t. x,

$$\frac{dy}{dx} = \phi\frac{du}{dx} + \psi\frac{dv}{dx} + u\frac{d\phi}{dx} + v\frac{d\psi}{dx}$$

Let us choose ϕ & ψ such that,

$$u\frac{d\phi}{dx} + v\frac{d\psi}{dx} = 0 \quad \text{——— } (5)$$

then, $\dfrac{dy}{dx} = \phi\dfrac{du}{dx} + \psi\dfrac{dv}{dx}$

$$\Rightarrow \frac{d^2y}{dx^2} = \phi\frac{d^2u}{dx^2} + \psi\frac{d^2v}{dx^2} + \frac{d\phi}{dx}\frac{du}{dx} + \frac{d\psi}{dx}\frac{dv}{dx}$$

Substituting the values of y, y', y'' in ①,

$$\phi\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)+\psi\left(\frac{d^2v}{dx^2}+P\frac{dv}{dx}+Qv\right)$$

$$+\frac{d\phi}{dx}\cdot\frac{du}{dx}+\frac{d\psi}{dx}\frac{dv}{dx}=X$$

$$\Rightarrow \frac{d\phi}{dx}\frac{du}{dx}+\frac{d\psi}{dx}\frac{dv}{dx}=X \quad \text{———— ⑥}$$

Solving for $\frac{d\phi}{dx}$ and $\frac{d\psi}{dx}$ from ⑤ & ⑥,

$$\frac{d\phi}{dx}=\frac{vX}{v\frac{du}{dx}-u\frac{dv}{dx}} \quad \& \quad \frac{d\psi}{dx}=-\frac{uX}{v\frac{du}{dx}-u\frac{dv}{dx}}$$

Integrating we get,

$$\phi=c_1+\int\frac{vX\,dx}{v\frac{du}{dx}-u\frac{dv}{dx}} \quad \& \quad \psi=c_2-\int\frac{uX\,dx}{v\frac{du}{dx}-u\frac{dv}{dx}}$$

Substituting these values of ϕ & ψ in ④, we get the complete soln of ① where c_1, c_2 are constants.

(1)
$$\frac{d^2y}{dx^2} + a^2 y = \sec ax \quad \text{——— (1)}$$

C.F. of (1) is $A\cos ax + B\sin ax$ where A, B are const coeffs.
Here $\cos ax$ & $\sin ax$ are indep. soln. of the corresponding homogeneous eqn of (1), since Wronskian,

$$W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & \cos ax \end{vmatrix} = a\cos^2 ax + a\sin^2 ax = a \neq 0.$$

Assume that ϕ & ψ are functions of x & choose in such a way that,

$y = \phi\cos ax + \psi\sin ax$ satisfies (1).

$$\Rightarrow \frac{dy}{dx} = -\phi a\sin ax + \psi a\cos ax + \cos ax\frac{d\phi}{dx} + \sin ax\frac{d\psi}{dx} \quad \text{——— (2)}$$

We choose ϕ & ψ such that,

$$\cos ax\frac{d\phi}{dx} + \sin ax\frac{d\psi}{dx} = 0 \quad \text{——— (3)}$$

$$\therefore \frac{dy}{dx} = -\phi a\sin ax + \psi a\cos ax$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\phi a^2\cos ax - \psi a^2\sin ax - a\sin ax\frac{d\phi}{dx} + a\cos ax\frac{d\psi}{dx}$$

$\therefore$ putting values of y, y', y'' in (1),

$$\phi' u' + \psi' v' = X$$

$$\Rightarrow -a\sin ax\frac{d\phi}{dx} + a\cos ax\frac{d\psi}{dx} = \sec ax \quad \text{——— (4)}$$

From (3) & (4), $a\frac{d\phi}{dx} = -\tan ax$ & $a\frac{d\psi}{dx} = 1$

Integrating, $\phi = \frac{1}{a^2}\log\cos ax + c_1$ & $\psi = \frac{x}{a} + c_2$

$\therefore$ Complete soln., $y = \cos ax\left(\frac{1}{a^2}\log\cos ax + c_1\right) + \sin ax\left(\frac{x}{a} + c_2\right)$

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② $y'' + 4y = 4\tan 2x$ —— ①

$y_{C.F.} = A\cos 2x + B\sin 2x$, where A & B are constants & $\cos 2x$, $\sin 2x$ are indep. soln., since their Wronskian is nonzero.

Assume that the complete soln. is,

$y = \phi\cos 2x + \psi \sin 2x$ —— ② where ϕ, ψ are func. of x.

$\therefore y' = -2\phi\sin 2x + 2\psi\cos 2x + \cos 2x\,\phi' + \sin 2x\,\psi'$.

Choose ϕ & ψ such that, $\cos 2x\cdot\phi' + \sin 2x\cdot\psi' = 0$ —— ③

$\therefore y' = -2\phi\sin 2x + 2\psi\cos 2x$

$\Rightarrow y'' = -4\phi\cos 2x - 4\psi\sin 2x - 2\phi'\sin 2x + 2\psi'\cos 2x$

$\therefore$ Putting value of y, y'' in ①,

$-2\sin 2x\,\phi' + 2\cos 2x\cdot\psi' = 4\tan 2x$

$\Rightarrow \cos 2x\cdot\psi' - \sin 2x\cdot\phi' = 2\tan 2x$ —— ④

Solving ③, ④, we get,

$\phi' = -\dfrac{2\sin^2 2x}{\cos 2x}$ & $\psi' = 2\sin 2x$

Integrating, $\phi = -2\displaystyle\int \frac{\sin^2 2x}{\cos 2x}dx = -2\int \frac{1-\cos^2 2x}{\cos 2x}dx$

$= \sin 2x - \log(\sec 2x + \tan 2x) + c_1$

& $\psi = 2\displaystyle\int \sin 2x\,dx = c_2 - \cos 2x$

$\therefore$ soln. $y = \cos 2x(\sin 2x - \log(\sec 2x + \tan 2x) + c_1)$
$+ \sin 2x(c_2 - \cos 2x)$.

(3) $$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x} \quad \text{——— (1)}$$

$$y_{C.F.} = Ae^x + Be^{-x}$$

Let the complete solu^n be, $y = \phi e^x + \psi e^{-x}$ ——— (2)

$$\therefore \frac{dy}{dx} = \phi e^x - \psi e^{-x} + e^x\phi' + e^{-x}\psi'$$

we choose ϕ, ψ such that $e^x\phi' + e^{-x}\psi' = 0$ ——— (3)

$$\therefore y' = \phi e^x - \psi e^{-x}$$

$$\therefore y'' = \phi e^x + \psi e^{-x} + e^x\phi' - e^{-x}\psi'$$

Putting the values of y, y'' in (1),

$$e^x\phi' - e^{-x}\psi' = \frac{2}{1+e^x} \quad \text{——— (4)}$$

Solving (3) & (4),

$$\frac{d\phi}{dx} = \frac{e^{-x}}{1+e^x} \quad \& \quad \frac{d\psi}{dx} = \frac{-e^x}{1+e^x}$$

Integrating,

$$\phi = \int \frac{e^{-x}}{1+e^x}dx = \int \frac{dz}{z^2(1+z)} \qquad \left[e^x = z\right.$$

$$= \int \left(\frac{1}{z^2} - \frac{1}{z} + \frac{1}{1+z}\right)dz = -\frac{1}{z} + \log\frac{1+z}{z} + c_1$$

$$= c_1 - e^{-x} + \log\frac{1+e^x}{e^x}$$

$$\psi = -\int \frac{e^x}{1+e^x}dx = c_2 - \log(1+e^x)$$

$$\therefore \text{Solu}^n,\ y = e^x\left(c_1 - e^{-x} + \log\frac{1+e^x}{e^x}\right) + e^{-x}\left(c_2 - \log(1+e^x)\right)$$