# Problems on The Riemann Integral

1. Let $f$ be continuous function on an interval $[a,b]$. Prove that
$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f(a+\frac{k}{n}(b-a)).$$
2. Let $f$ be continuous function on an interval $[a,b]$. Prove that
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=a(n)}^{b(n)} f(\frac{k}{n}) = \int_{\lim\limits_{n\to\infty} \frac{a(n)}{n}}^{\lim\limits_{n\to\infty} \frac{b(n)}{n}} f(x)dx.$$
3. Prove that a bounded real valued function $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ iff for every $\epsilon>0$, there exists a partition $\mathscr{P}$ of $[a,b]$ such that $\mathscr{U}(\mathscr{P},f)-\mathscr{L}(\mathscr{P},f)<\epsilon$. Is this result true for any refinement of $\mathscr{P}$? Is this result true for unbounded functions? Justify.
4. If $f\in\mathcal{R}[a,b]$, prove that $f^2\in\mathcal{R}[a,b]$. Is the converse true? Justify.
5. Let $f$ be a bounded function on $[a,b]$ and let $D=\{x\in[a,b]: f \textit{ is discontinuous at } x\}$. Prove that if $D$ has content zero, then $f\in\mathcal{R}[a,b]$.
6. Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Prove that $f\in\mathcal{R}[a,b]$ iff it is continuous almost everywhere.
7. Let $f\in\mathcal{R}[a,b]$ and $g:[c,d]\to[a,b]$ be a $C^1$-diffeomorphism. Prove that $f\circ g\in\mathcal{R}[c,d]$.
8. Let $f\in\mathcal{R}[a,b]$ be bounded with Range$(f)\subset[c,d]$. If $g:[c,d]\to\mathbb{R}$ be continuous, then prove that $g\circ f\in\mathcal{R}[a,b]$.
9. Check the Riemann integrability of $f$, where $f:[a,b]\to\mathbb{R}$ by

   (i) $f(x)=\begin{cases} x^2 & ,x\in\mathbb{Q}\cap[a,b] \\ x^3 & ,x\in\mathbb{Q}^c\cap]a,b] \end{cases}$ (ii) $f(x)=\begin{cases} a & ,x\in[0,\frac{a+b}{2}) \\ \frac{a+b}{2} & ,x=\frac{a+b}{2} \\ b & ,x\in(\frac{a+b}{2},b] \end{cases}$, where $0<a<b$

   (iii) $f(x)=\begin{cases} 0 & ,x\in\mathbb{Q}^c\cap[a,b] \\ \frac{1}{q} & ,x=\frac{p}{q} \textit{ in } [a,b] \textit{ with } p\in\mathbb{Z},\ q\in\mathbb{N};\ p,q \textit{ have no common factor.} \end{cases}$

   (iv) $f(x)=\begin{cases} sin(\frac{1}{x}) & ,x\neq 0 \\ 0 & ,x=0 \end{cases}$, where $[a,b]=[-1,1]$

   (v) $f(x)=\begin{cases} x\ sgn(sin\frac{1}{x}) & ,x\neq 0 \\ 0 & ,x=0 \end{cases}$, where $[a,b]=[-1,1]$ and $sgn$ represents the signum function.

   (vi) $f(x)=\begin{cases} 1 & ,\textit{if } x \textit{ is algebric} \\ 0 & ,\textit{otherwise} \end{cases}$, where $[a,b]=[0,1]$

   (vii) $f(x)=\lim\limits_{n\to\infty} sin^{2n}(24\pi x)$,where $[a,b]=[0,1]$

   Also Find $\mathscr{U}(\mathscr{P},f)$, $\mathscr{L}(\mathscr{P},f)$ for some partition $\mathscr{P}$ of $[a,b]$ and both upper, lower integrals.

10. Let $\{r_1,r_2,...,r_n,...\}$ be an enumeration of the rationals in the interval $[0,1]$. Define, for each $n\in\mathbb{N}$ and for each $x\in[0,1]$, $f_n(x)=\begin{cases} 1 & ,if \quad x=r_1,r_2,...,r_n \\ 0 & ,otherwise \end{cases}$, Check the Riemann integrability of $f_n(x)$.
11. Prove that every Riemann integrable function is Darboux integrable and conversely.
12. Let $f:[a,b]\to\mathbb{R}$ be continuous and non-negative. If $\int_a^b f(x)dx=0$ then prove that $f(x)=0$ for all $x\in[a,b]$. What happen if the continuity of $f$ been removed?
13. Let $f:[a,b]\to\mathbb{R}$ be continuous. Suppose that for every $g:[a,b]\to\mathbb{R}$ with $g\in\mathcal{R}[a,b]$, the product $fg$ is Riemann integrable and $\int_a^b(fg)(x)dx=0$. Prove that $f(x)=0$ for all $x\in[a,b]$.
14. Let $f$ be bounded real valued function on $[a,b]$. If $\lim\limits_{||\mathscr{P}||\to 0}\mathscr{S}(\mathscr{P},f)=I$, then prove that $f\in\mathcal{R}[a,b]$ with $\int_a^b f(x)dx=I$ and conversely,where $\mathscr{S}(\mathscr{P},f)$ is the Riemann sum.
15. Let $f$ be continuously differentiable real valued function on $[a,b]$ such that, $|f'(x)|\le k\ \ \forall\ x\in[a,b]$. For a partition $\mathscr{P}=\{a=x_0,x_1,\ldots,x_n=b\}$. Check whether $\mathscr{U}(\mathscr{P},f)-\mathscr{L}(\mathscr{P},f)\le k||\mathscr{P}||(b-a)$ or not, where $||\mathscr{P}||$ is the maximum length of the subinterval $[x_{k-1},x_k]$, $k=1,2,\ldots,n$.
16. Let $f:[\frac{1}{2},2]\to\mathbb{R}$ be a strictly increasing function and $g(x)=f(x)+f(\frac{1}{x})$, $x\in[1,2]$. Then prove that for any partition $\mathscr{P}$ of $[1,2]$, $\mathscr{U}(\mathscr{P},g)\ge\mathscr{L}(\mathscr{P},g)$, for all choice of $f$.

17. Let $f$ be a monotone function on $[0,1]$, defined by $f(x)=\sum_{n=1}^{\infty}\frac{1}{2^n}I(x-x_n)$, where $I:\mathbb{R}\to\mathbb{R}$ be unit jump function and $x_n=\frac{n}{n+1}$, $n\in\mathbb{N}$. Find $\int_0^1 f(x)dx$.
18. Let $f$, $g$, $h$ be bounded real-valued functions on $[a,b]$ satisfying $f(x)\le g(x)\le h(x)$ for all $x\in[a,b]$ with $\int_a^b f(x)dx=\int_a^b h(x)dx=I$, Prove that $g\in\mathcal{R}[a,b]$ with $\int_a^b g(x)dx=I$.
19. If $f\in\mathcal{R}[a,b]$ and if $F$ is an anti-derivative of $f$ on $[a,b]$, prove that $\int_a^b f(x)dx=F(b)-F(a)$.
20. Let $f\in\mathcal{R}[a,b]$. Define $F$ on $[a,b]$ by $\int_a^b f(x)dx=F(x)$. Prove that $F$ is continuous on $[a,b]$. Furthermore, if $f$ is continuous at a point $c\in[a,b]$, then $F$ is differentiable at $c$ and $F'(c)=f(c)$. Does integrability of $f$ always imply the existence of an anti-derivative of $f$?
21. Let $f$ be continuous real valued function on $[a,b]$. Prove that there exists $c\in[a,b]$ such that $\int_a^b f(x)dx=f(c)(b-a)$. What happen if the continuity of $f$ been removed? In that case find the supremum and infimum for the integral of $f$.
22. Let $f$, $g$ be differentiable functions on $[a,b]$ with $f'$, $g'\in\mathcal{R}[a,b]$. Prove that $\int_a^b f(x)g'(x)dx=f(b)g(b)-f(a)g(a)-\int_a^b f'(x)g(x)dx$.
23. Let $\phi$ be differentiable on $[a,b]$ with $\phi'\in\mathcal{R}[a,b]$. If $f$ is continuous on $\phi([a,b])$,then prove that $\int_a^b f(\phi(t))\phi'(t)dt=\int\limits_{\phi(a)}^{\phi(b)} f(x)dx$.
24. Suppose $f:[a,b]\to\mathbb{R}$ is continuous and $g,h:[c,d]\to[a,b]$ are differentiable. For $x\in[c,d]$ define $H(x)=\int\limits_{h(x)}^{g(x)} f(t)dt$. Find $H'(x)$. What can you say about the continuity, uniform continuity and differentiability of both $H(x)$ and $H'(x)$?
25. Suppose $f:[a,b]\to\mathbb{R}$ is continuous real valued function and define $H$ on $[a,b]$ by $H(x)=\int\limits_a^b f(x)dx$. Find $H'(x)$.
26. Let $f$ be a continuous real valued function on $[a,b]$, $g\in\mathcal{R}[a,b]$ with $g(x)\ge 0$ for all $x\in[a,b]$. Prove that there exists $c\in[a,b]$ such that $\int_a^b f(x)g(x)dx=f(c)\int_a^b g(x)dx$. Also discuss the case when $f$ is increasing and decreasing on $[a,b]$ respectively.
27. Let $\phi$ be a real valued differentiable function on $[a,b]$ with $\phi'(x)\neq 0$ for all $x\in[a,b]$. Let $\psi$ be the inverse function of $\phi$ on $I=\phi([a,b])$. If $f:I\to\mathbb{R}$ is continuous on $I$,then prove that $\int_a^b f(\phi(x))dx=\int\limits_{\phi(a)}^{\phi(b)} f(t)\psi'(t)dt$.
28. Suppose $f$ is continuous on $[0,1]$. Prove that $\lim\limits_{n\to\infty}\int_0^1 f(x^n)dx=f(0)$. Also prove that if $M=max\{\mid f(x)\mid : x\in[a,b]\}$, then $\lim\limits_{n\to\infty}\left(\int_0^1 \mid f(x)\mid^n dx\right)^{\frac{1}{n}}=M$.
29. Let $f$ be increasing function on $[a,b]$ and let $F(x)=\int\limits_a^x f(t)dt$. Prove that $F_+^{'}(c)=f(c+)$ and $F_-^{'}(c)=f(c-)$ for every $c\in(a,b)$.
30. Let $f,g\in\mathcal{R}[a,b]$. Prove that $\left|\int_a^b f(x)g(x)dx\right|^2\le\left(\int\limits_a^b f^2(x)dx\right)\left(\int\limits_a^b g^2(x)dx\right)$.
31. Let $\varphi$ be a non-negative and continuous function on $[0,\infty[$ and such that $\varphi(x)\le\int_0^x\varphi(t)dt$, for all $x\in[0,\infty[$. Prove that $\varphi\equiv 0$.
32. Let $f$ be a strictly increasing continuous function mapping $[0,1]$ onto $[0,1]$. Give a geometric argument showing $\int_0^1 f(x)\,dx+\int_0^1 f^{-1}(u)\,du=1$.
33. Prove that $\lim\limits_{x\to 0}\dfrac{\int_{-x}^{x} f(t)dt}{\int_0^{2x} f(t+1)dt}=\dfrac{f(0)}{f(1)}$, where $f$ is continuous on $\mathbb{R}$.
34. Evaluate:

(i) $\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{k=1}^{[\frac{n}{2}]} cos\left(\dfrac{k\pi}{n}\right)$ (ii) $\lim\limits_{n\to\infty}\left(\dfrac{\sum\limits_{r=1}^{k}(n+r)^m}{n^{m-1}}-kn\right)$, where $m$, $k$ be fixed positive integers.

(iii) $\lim\limits_{x\to 0}\dfrac{\int_0^{x^2} e^{\sqrt{1+t}}\,dt}{x^2}$ (iv) $\lim\limits_{x\to 0}\dfrac{\int_0^x e^{t^2}\,dt}{x}$ (v) $\lim\limits_{x\to 0}\dfrac{\int_0^{x^2} sin\sqrt{t}\,dt}{x^3}$ (vi) $\displaystyle\int_{-1}^{1}\frac{e^{2\,tan^{-1}t}}{1+t^2}dt$

(vii) $\frac{d}{dx}\left(\int_0^x(2x-t)^n f(t)dt\right)$, where $f:\mathbb{R}\to\mathbb{R}$ is continuous and $n\in\mathbb{N}$ (viii) $\frac{d}{dx}\left(\int_{-x}^{x}\frac{1-e^{-xy}}{y}dy\right)$

(ix) $\frac{d}{dx}\left(\int_{x^2}^{x^3} tan(xy^2)dy\right)$, where $x>1$
35. Prove the following inequalities by mean value theorem:

(i) $\frac{\pi^3}{96} < \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2}{5+3\,sin\,x} dx < \frac{\pi^3}{24}$ (ii) $-\frac{1}{2} < \int_0^1 \frac{x^3\,cos\,5x}{2+x^2} dx < \frac{1}{2}$