Problems on Basic Topology on $\mathbb{R}$

Problems on Interior point, cluster points, isolated points and boundary points

1. Determine the set of interior points, cluster points, isolated points, exterior points and boundary points for each of the following sets:
   (i) $S=$ any finite set, (ii) $S=\{p+\frac{1}{n}:n\in\mathbb{N},p\in\mathbb{R},p\ is\ fixed\}$, (iii) $S=\mathbb{R}\setminus\mathbb{Z}$, (iv) $S=(0,1]\setminus\{\frac{1}{2^n}:n\in\mathbb{N}\cup\{0\}\}$, (v) $S=\{sin(\frac{1}{n}):n\in\mathbb{N}\}$, (vi) $S=\{nsin(\frac{1}{n}):n\in\mathbb{N}\}$, (vii) $S=\{(-1)^n(1-\frac{1}{2^n}):n\in\mathbb{N}\}$, $(viii)$ $S=\{\frac{1}{m}+\frac{1}{n}:m,n\in\mathbb{N}\}$, (ix) $S=\{\frac{(1)^n}{m}+\frac{1}{n}:m,n\in\mathbb{N}\}$, (x) $S=\{\frac{(-1)^n}{m}+\frac{(-1)^m}{n}:m,n\in\mathbb{N}\}$
2. Give an example of each of the following or explain why you think such a set could not exist.
   (i) A non-empty set with no cluster points and no isolated points.
   (ii) A non-empty set with no interior points and no isolated points.
   (iii) A non-empty set with no boundary points and no isolated points.
   (iv) A non-empty set with no cluster points and no boundary points.
   (v) A non-empty set with no interior points and no boundary points.
3. Show that every interior point of a set must also be an accumulation point of that set, but not conversely.
4. Let A be a set and $B=\mathbb{R}\setminus A$.(i)Show that every boundary point of A is also a boundary point of B.
   (ii) Let A be a set and $B=\mathbb{R}\setminus A$.Show that every boundary point of A is a point of accumulation of A or else a point of accumulation of B, perhaps both.
   (iii) Show that every interior point of A is not an accumulation point of B.
   (iv) Show that every accumulation point of A is not an interior point point of B.
5. Show that every accumulation point of a set that does not itself belong to the set must be a boundary point of that set.
6. Show that a point x is not an interior point of a set E if and only if there is a sequence of points $\{x_n\}$ converging to x and no point $x_n\in E$.
7. Prove that every neighbourhood of any limit point of a set contains infinitely many elements of that set.
8. Show that there is no set which has the set $\mathbb{Q}$ as its only set of accumulation points.
9. Show that there is no set with uncountably many isolated points.
10. Suppose that $\{x_n\}$ is a convergent sequence converging to a number L and that $x_n\neq L$ for all n.Show that the set $\{x:x=x_n\ for\ some\ n\}$ has exactly one point of accumulation, namely L.Of what importance was the assumption that $x_n\neq L$ for all n for this exercise?

Problems on open and closed sets

1. Is it true that a set, all of whose points are isolated, must be closed?
2. If a set has no isolated points must it be closed? Must it be open?
3. If a set contains all of its interior points then the set is open. Justify the statement.
4. Determine which of the following sets are open, which are closed, and which are neither open nor closed,considered in the beginning of the assignment.
5. Show that the closure operation has the following properties:
   (i) if $E_1\subset E_2$, then $\overline{E_1}\subset\overline{E_2}$, (ii) $\overline{\overline{E}}=\overline{E}$, (iii) $\overline{E_1\cup E_2}=\overline{E_1}\cup\overline{E_2}$. Is the result true for intersection?
6. Show that the interior operation has the following properties:
   (i) if $E_1\subset E_2$, then $int(E_1)\subset int(E_2)$ (ii) $int(int(E))=int(E)$ (iii) $int(E_1\cap E_2)=int(E_1)\cap int(E_2)$. Is the result true for union?
7. Show that if the derived set $E^{'}$ of a set E is empty then the set E is closed.
8. Show that the derived set $E^{'}$ of any set E is closed.
9. Show that the set of all interior points $int(E)$ of any set E is open.
10. Show that the set E is (i) open iff $int(E)=E$, (ii) closed iff $\overline{E}=E$

Problems on Elementary Topology

1. For any two sets A and B
   (i) If A is open and B is closed, what can you say about the sets AB, $A\setminus B$ and $B\setminus A$.
   (ii) If both A and B are open, what can you say about the sets AB, $A\setminus B$ and $B\setminus A$.
   (iii) If both A and B are closed, what can you say about the sets AB, $A\setminus B$ and $B\setminus A$.
   (iv) If A and B are both open and closed, what can you say about the sets AB, $A\setminus B$ and $B\setminus A$. Generalise the case when $A=\mathbb{R}$.
2. Prove that only $\emptyset$ and $\mathbb{R}$ are clopen subsets of $\mathbb{R}$.
3. Prove that arbitrary union of open sets is again open. Is the result true for arbitrary intersection?
4. Prove that arbitrary intersection of closed sets is again closed. Is the result true for arbitrary union?
5. Show that the set $int(E)$ can be described as the largest open set that is contained inside E.
6. Show that the set $\overline{E}$ can be described as the smallest closed set that contains every point of E.
7. Prove that a non-empty bounded open set in $\mathbb{R}$ can be represented as the union of a countable collection of disjoint open intervals.
8. (i)Express the closed interval $[0,1]$ as an intersection of a sequence of open sets. Can it also be expressed as

a union of a sequence of open sets?

(ii) Express the open interval $(0, 1)$ as a union of a sequence of closed sets. Can it also be expressed as an intersection of a sequence of closed sets?

9. Prove that set of real numbers E is closed and bounded if and only if every infinite subset of E has a point of accumulation that belongs to E.
10. Suppose that a function f is locally bounded at each point of a closed and bounded set E. Then prove that f is bounded on the whole of the set E.
11. Define nested intervals. State and prove Cantor's intersection theorem.
12. Prove that a set $S \subset \mathbb{R}$ is everywhere dense iff it intersects every non-empty open sets in $\mathbb{R}$.
13. Verify that a set A is said to be dense in a set B iff $B \subset \overline{A}$. Generalise the case when $A = B$
14. (i) Is $\mathbb{R} \setminus \mathbb{Q}$ deense in $\mathbb{Q}$? (ii) Is $A = \{x : x = \frac{m}{2^n}, m \in \mathbb{Z}, n \in \mathbb{N}\}$ dense in $\mathbb{Q}$? (iii) Is $A = \{p + q\sqrt{2} : p, q \in \mathbb{Q}\}$ dense in $\mathbb{R} \setminus \mathbb{Q}$?
15. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous map. Does f maps dense sets to dense sets? Does f maps nowhere dense sets to nowhere dense sets?
16. Prove that the complement of a dense open subset of $\mathbb{R}$ is nowhere dense in $\mathbb{R}$.
17. If a set A is nowhere dense, what can you say about $\mathbb{R} \setminus A$? What happen if A is dense in $\mathbb{R}$?
18. Give an example of a sequence of nowhere dense sets whose union is not nowhere dense.
19. Define residual set. Prove that every residual subset of $\mathbb{R}$ is dense in $\mathbb{R}$.