Problems on Numerical Sequences

Problems on monotone sequences

1. Show that the sequence $\{u_n\}$ defined by $u_1 = 0,\ u_2 = \frac{1}{2},\ u_{n+1} = \frac{1}{3}(1 + u_n + u_{n-1}^3),\ n \in \mathbb{N}$ converges and determine its limit.
2. For a fixed positive number α and fixed natural number p, if $u_1 > \sqrt{\alpha}$, and $u_{n+1} = \frac{p-1}{p}u_n + \frac{\alpha}{p}u_n^{-p+1},\ n \in \mathbb{N}$, then describe the behaviour of $\{u_n\}$.
3. For a fixed positive number α, if $u_1 > \sqrt{\alpha}$, and $u_{n+1} = \frac{\alpha+u_n}{1+u_n},\ n \in \mathbb{N}$, then check the monotonicity of $\{u_{2m}\}$ and $\{u_{2m-1}\}$. Also check the convergence of $\{u_n\}$.
4. Let $a_1, a_2, ..., a_p$ be fixed positive numbers. Consider the sequence $\{u_n\}$ by $u_n = \sqrt[n]{\frac{a_1^n+a_1^n+a_2^n+...+a_p^n}{p}},\ n \in \mathbb{N}$. Show that $\{u_n\}$ is monotone increasing.
5. Let f, g be continuous and positive functions defined on $[0, 1]$. Suppose that $\int_0^1 f(x)dx = \int_0^1 g(x)dx$ and for every integer $n \geq 0$, define $y_n = \int_0^1 \frac{(f(x))^{n+1}}{(g(x))^n}dx$. Check the monotonicity of y_n.
6. if $\{u_n\}$ be a bounded sequence such that $u_{n+1} > u_n - \frac{1}{2^n},\ n \in \mathbb{N}$. Show that $\{u_n\}$ is convergent.
7. Check the convergence of the sequence $\{u_n\}$, where
   $(i)\ u_n = -2\sqrt{n} + (1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}})$
   $(ii)\ u_n = -2\sqrt{n+1} + (1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}})$.
8. For $c > 2$, define $\{u_n\}$ by $u_1 = c^2$ and $u_{n+1} = (u_n - c)^2,\ n \in \mathbb{N}$. Show that $\{u_n\}$ is strictly increasing.
9. Let a be arbitrarily fixed and let c be defined as follows: $u_1 \in \mathbb{R}$ and $u_{n+1} = u_n^2 + (1 - 2a)u_n + a^2,\ n \in \mathbb{N}$. Determine all u_1 such that the sequence $\{u_n\}$ converges and in such a case find limit of $\{u_n\}$.
10. Show the convergence and find the limit of the sequence $\{u_n\}$, where $u_n = \frac{n+1}{2^{n+1}}(2 + \frac{2^2}{2} + ... + \frac{2^n}{n}),\ n \in \mathbb{N}$.
11. Let $\{u_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} \left| u_n + 3\left(\frac{n-2}{n}\right)^n \right|^{\frac{1}{n}} = \frac{3}{5}$. Find $\lim_{n\to\infty} u_n$.
12. Find the limit of the sequence $\{u_n\}$, where
    $(i)\ u_n = (\sqrt{2} - \sqrt[3]{2})(\sqrt{2} - \sqrt[5]{2})...(\sqrt{2} - \sqrt[2n+1]{2})$
    $(ii)\ u_n = (\frac{1}{1.2.3} + \frac{1}{2.3.4} + ... + \frac{1}{n.(n+1).(n+2)})$
    $(iii)\ u_n = \frac{1.1!+2.2!+3.3!+...+n.n!}{(n+1)!}$
    $(iv)\ u_n = \frac{n!}{(2n+1)!!}$
    $(v)\ u_n = \frac{a+aa+aaa+...+a...a(ntimes)}{10^n}$, where $a \in \{1, 2, ..., 9\}$
    $(vi)\ u_n = \frac{1}{n}\left[(n + 1)(n + 2)...(n + n)\right]^{\frac{1}{n}}$

Problems on limit point, limit superior, limit infirior

1. Determine the set of all limit points, limit superior, limit inferior of $\{u_n\}$, where
   $(i)\ u_n = \frac{2n^2}{7} - [\frac{2n^2}{7}]$
   $(ii)\ u_n = n^{(-1)^n n}$
   $(iii)\ u_n = n\alpha - [n\alpha]$, where α is real
   $(iv)\ u_n = sin(n\pi\alpha)$, where α is real
   $(v)\ u_n = (1 + \frac{(-1)^n}{n})^n + sin(\frac{n\pi}{4})$
2. Find the upper and lower limit of the sequence $\{u_n\}$, defined by $u_1 = 0,\ u_{2m} = \frac{u_{2m-1}}{2},\ u_{2m+1} = \frac{1}{2} + u_{2m}$.
3. For any two sequences $\{u_n\}$ and $\{v_n\}$, prove that
   $$\lim_{n\to\infty} inf(u_n) + \lim_{n\to\infty} inf(v_n) \leq \lim_{n\to\infty} inf(u_n + (v_n) \leq \lim_{n\to\infty} inf(u_n) + \lim_{n\to\infty} sup(v_n)$$
   $$\leq \lim_{n\to\infty} sup(u_n + v_n) \leq \lim_{n\to\infty} sup(u_n) + \lim_{n\to\infty} sup(v_n)$$
   excluding the indeterminate forms of type $\infty - \infty$. What happen if the additions in above inequalities are replaced by multiplication?
4. Prove that for any positive sequence $\{u_n\}$,
   $$\lim_{n\to\infty} inf(\frac{u_{n+1}}{u_n}) \leq \lim_{n\to\infty} inf(\sqrt[n]{u_n}) \leq \lim_{n\to\infty} sup(\sqrt[n]{u_n}) \leq \lim_{n\to\infty} sup(\frac{u_{n+1}}{u_n})$$
5. For any two sequences $\{u_n\}$ and $\{v_n\}$, prove that
   $$\lim_{n\to\infty} sup(max\{u_n, v_n\}) = max\{\lim_{n\to\infty} sup(u_n), \lim_{n\to\infty} sup(v_n)\}$$

and

$$\lim_{n\to\infty} inf(max\{u_n, v_n\}) = max\{\lim_{n\to\infty} inf(u_n), \lim_{n\to\infty} inf(v_n)\}$$

Are the above results also holds for minimum? Justify.

6. Prove that every bounded sequence of real numbers contains a convergent sub-sequence.
7. Let $\{u_n\}$ be a sequence of real numbers. Then $\{u_n\}$ is convergent if and only if $\lim_{n\to\infty} inf(u_n) = \lim_{n\to\infty} sup(u_n) = \lim_{n\to\infty}(u_n)$ and these are finite.
8. What relation, if any, can you state for the limit superior and limit inferior of a sequence $\{u_n\}$ and one of its sub-sequences $\{u_{n_k}\}$?
9. If a sequence $\{u_n\}$ has no convergent sub-sequences, what can you state about the limit superior and limit inferior of the sequence?
10. Let S denote the set of all real numbers t with the property that some subsequence of a given sequence $\{u_n\}$ converges to t. What is the relation between the set S and the limit superior and limit inferior of the sequence $\{u_n\}$?
11. For any sequence $\{u_n\}$, write $s_n = \frac{u_1+u_2+...+u_n}{n}$. Show that

$$\lim_{n\to\infty} inf(u_n) \le \lim_{n\to\infty} inf(s_n) \le \lim_{n\to\infty} sup(s_n) \le \lim_{n\to\infty} sup(u_n)$$

Give an example to show that each of these inequalities may be strict.

Problems on Cauchy sequences

1. Define Cauchy sequences. Prove that a real sequence is convergent if and only if it is a Cauchy sequence. Show also that every subsequence of a Cauchy sequence is Cauchy.
2. State and prove the necessary and sufficient condition for the convergence of a real sequence.
3. Show that any multiple of a Cauchy sequence is again a Cauchy sequence.
4. Prove or disprove that if for a sequence $\{u_n\}$, for all $\epsilon > 0$ there exists an integer N with the property that $\mid u_{n+1} - u_n \mid < \epsilon$, whenever $n \ge N$, then the sequence is Cauchy sequence.
5. What can you conclude about the sequence $\{u_n\}$, if there exists $\epsilon > 0$ be such that for all positive integer N with the property that $\mid u_m - u_n \mid < \epsilon$, whenever $m, n \ge N$.
6. Show that if $\lim_{n\to\infty} u_n = l$, then $\lim_{n\to\infty} \dfrac{u_1 + u_2 + ... + u_n}{n} = l$. Is the converse true?
7. Show that if $\lim_{n\to\infty} u_n = l$, then $\lim_{n\to\infty} \dfrac{n.u_1 + (n-1).u_2 + ... + 1.u_n}{n^2} = \dfrac{l}{2}$. Is the converse true?
8. Show that if $\lim_{n\to\infty} u_n = l$, then $\lim_{n\to\infty} \sqrt[n]{u_1 u_2 ... u_n} = l$. Is the converse true?
9. For a positive sequence $\{u_n\}$, show that if $\lim_{n\to\infty} \dfrac{u_{n+1}}{u_n} = l$, then $\lim_{n\to\infty} \sqrt[n]{u_n} = l$. Is the converse true?
10. Let $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$. Show that $\lim_{n\to\infty} \dfrac{u_1 v_n + u_2 v_{n-1} + ... + u_n v_1}{n} = uv$.
11. Find the limit of the sequence $\{u_n\}$ where $(i)\ u_n = \frac{n}{a^{n+1}} \displaystyle\sum_{r=1}^{n} \frac{a^r}{r}$, where $a > 1$ $(ii)\ u_n = \frac{1}{n^{k+1}} \displaystyle\sum_{r=0}^{n} \frac{(k+r)!}{r!}$, where $k \in \mathbb{N}$. $(iii)\ u_n = \frac{1}{\sqrt{n}} \displaystyle\sum_{r=0}^{n} \frac{1}{\sqrt{n+r}}$, where $a > 1$ $(iv)\ u_n = \dfrac{1 + \sum_{r=1}^{n} r.a^r}{n.a^{n+1}}$, where $a > 1$ $(v)\ u_n = \frac{1}{\sqrt{n}} \displaystyle\sum_{r=1}^{n} \frac{a_r}{\sqrt{r}}$, when $\{a_n\}$ converges to a. $(vi)\ u_n = \displaystyle\sum_{r=0}^{n-1} \frac{a_{n-r}}{2^r}$, when $\{a_n\}$ converges to a. $(vii)\ u_n = \displaystyle\sum_{r=1}^{n} \frac{a_{n+1-r}}{r.(r+1)}$, when $\{a_n\}$ converges to a. $(viii)\ u_n = \frac{\sqrt[k]{n}}{\sqrt[n]{n!}}$, $k \in \mathbb{N}$. (ix) $u_n = \sqrt[n]{\binom{nk}{n}}$, when k be a fixed integer greater than 1. (x) $u_n = n^x (a_1 a_2 ... a_n)^{\frac{1}{n}}$, when $\{n^x a_n\}$ converges to a for some real x.
12. Prove that if $\{u_n\}$ is a sequence for which $\lim_{n\to\infty}(u_{n+1} - u_n) = u$, then prove that $\lim_{n\to\infty} \dfrac{u_n}{n} = u$.
13. Suppose $\{u_n\}$ be such that the sequence $\{v_n\}$ with $v_n = 2u_n + u_{n-1}$, $n \ge 2$, converges to v. Study the convergence of $\{u_n\}$.
14. Calculate $(i)\ \lim_{n\to\infty}(n!e - [n!e])$ $(ii)\ \lim_{n\to\infty} n sin(2\pi n!e)$. $(iii)\ \lim_{n\to\infty} sin\left(\left(2n\pi + \dfrac{1}{2n\pi}\right) sin\left(2n\pi + \dfrac{1}{2n\pi}\right)\right)$ $(iv)\ \lim_{n\to\infty} \displaystyle\sum_{k=0}^{n} \binom{n}{k} sin\left(x + \frac{k\pi}{2}\right)$ $(v)\ \lim_{n\to\infty} \displaystyle\sum_{k=2}^{n} \frac{(n-1)!}{(n-k)!(k-2)!}$