Mathematical Economics
Semester-II
Paper-C-4
Some problems and their solutions

Prob. 1. If a firm has the production function $Q = L^{0.4}K^{0.6}$ and factor prices are constants, then show that the firm's long run marginal cost equals long run average cost.

Solution: The given production is $Q = L^{0.4}K^{0.6}$ where 'L' and 'K' units of labour and capital are used respectively to produce output level Q.

Suppose 'w' and 'r' are the constant unit prices of 'L' and 'K' respectively. If 'C' be the total cost of production then total cost equation becomes:

$$C = wL + rK$$

Now the aim of the producer is to minimize cost $C = wL + rK$ subject to output constraint $\bar{Q} = L^{0.4}K^{0.6}$.

The Lagrangian function to solve the problem becomes:

$Z = wL + rK + \lambda(\bar{Q} - L^{0.4}K^{0.6})$ where 'λ' denotes Lagrangian multiplier.

Hence first order condition of cost minimization require:

$$Z_L = \frac{\partial Z}{\partial L} = w - 0.4\lambda L^{-0.6}K^{0.6} = 0 \quad (1)$$

$$Z_K = \frac{\partial Z}{\partial K} = r - 0.6\lambda K^{-0.4}L^{0.4} = 0 \quad (2)$$

and $$Z_\lambda = \frac{\partial Z}{\partial \lambda} = \bar{Q} - L^{0.4}K^{0.6} = 0 \quad (3)$$

Now from (1) we get

$$0.4\lambda L^{-0.6}K^{0.6} = w \quad (4)$$

and from (2) we get

$$0.6\lambda K^{-0.4}L^{0.4} = r \quad (5)$$

Hence (4) ÷ (5) gives

$$\frac{0.4K^{0.6+0.4}}{0.6L^{0.4+0.6}} = \frac{w}{r}$$

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This implies

$$\frac{0.4K}{0.6L} = \frac{w}{r}$$

$$\text{or } K = \frac{0.6}{0.4}\cdot\frac{w}{r}L \quad (6)$$

Substituting equation (6) into the total cost equation we have:

$$C = wL + \frac{0.6}{0.4}wL$$

$$\Rightarrow C = \frac{wL}{0.4} \text{ or } L = \frac{0.4C}{w} \quad (7)$$

Again, substituting equation (7) into equation (6) we get,

$$K = \frac{0.6}{0.4}\cdot\frac{w}{r}\cdot\frac{0.4C}{w}$$

$$\Rightarrow K = \frac{0.6C}{r} \quad (8)$$

Let us now substitute equations (7) and (8) into the production function and we get:

$$Q = \left(\frac{0.4C}{w}\right)^{0.4}\left(\frac{0.6C}{r}\right)^{0.6}$$

$$\Rightarrow Q = \left(\frac{0.4}{w}\right)^{0.4}\left(\frac{0.6}{r}\right)^{0.6}C$$

$$\Rightarrow C = \frac{Q}{\left(\frac{0.4}{w}\right)^{0.4}\left(\frac{0.6}{r}\right)^{0.6}}$$

This is the long run total cost function.

Now long run average cost (LAC)

$$= \frac{C}{Q} = \frac{1}{\left(\frac{0.4}{w}\right)^{0.4}\left(\frac{0.6}{r}\right)^{0.6}}$$

Again, long run marginal cost (LMC)

$$= \frac{dC}{dQ} = \frac{1}{\left(\frac{0.4}{w}\right)^{0.4}\left(\frac{0.6}{r}\right)^{0.6}}$$

This shows that the firm's long run average cost (LAC) equals long run marginal cost (LMC).

Provided that the second order condition of cost minimization is satisfied.

Prob. 2. If MRS of y for x is $\frac{\alpha}{\beta} \cdot \frac{y+b}{x+a}$, show that one form of the individuals' utility function is $U = (x+a)^\alpha (y+b)^\beta$ where a, b, α and β are given constants.

Solution: Given that

$$\text{MRS of } y \text{ for } x = \frac{\alpha}{\beta} \cdot \frac{y+b}{x+a}$$

This implies

$$-\frac{dy}{dx} = \frac{\alpha}{\beta} \cdot \frac{y+b}{x+a} \quad \text{[As MRS is nothing but negative slope of IC]}$$

$$\Rightarrow \frac{\beta\, dy}{y+b} = \frac{\alpha\, dx}{x+a}$$

Integrating both sides we get,

$$\beta \int \frac{dy}{y+b} = -\alpha \int \frac{dx}{x+a}$$

$$\text{or } \beta \log(y+b) = -\alpha \log(x+a) + \log \bar{U}$$

where $\log \bar{U}$ denotes integrating constant

This implies

$$\log (x+a)^\alpha + \log (y+b)^\beta = \log \bar{U}$$

$$\text{or } \log\{(x+a)^\alpha (y+b)^\beta\} = \log \bar{U}$$

$$\text{or } \bar{U} = (x+a)^\alpha (y+b)^\beta \text{ where } \bar{U} \text{ is constant } U.$$

This is the form of indifference curve. So that one form of individuals utility function is

$$U = (x+a)^\alpha (y+b)^\beta$$

where U varies if quantities of x and y vary.

Prob. 3 Consider the production function $q = x_1^\alpha x_2^{1-\alpha}$ with $0 < \alpha < 1$. If input prices are r_1 and r_2 respectively then show that the expansion path is given by

$$(1-\alpha) r_1 x_1 - \alpha r_2 x_2 = 0$$

Solution: We know that along the expansion path

$$MRTS_{x_1, x_2} = \frac{r_1}{r_2}$$

where $MRTS_{x_1,x_2} = \frac{MP_{x_1}}{MP_{x_2}}$

Now $MP_{x_1} = \frac{\partial q}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha}$

and $MP_{x_2} = \frac{\partial q}{\partial x_2} = (1-\alpha) x_1^\alpha x_2^{-\alpha}$

Therefore,

$$MRTS_{x_1,x_2} = \frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) x_1^\alpha x_2^{-\alpha}}$$

$$\text{or } MRTS_{x_1,x_2} = \frac{\alpha}{1-\alpha} \cdot \frac{x_2}{x_1}$$

So that the equation of expansion path becomes:

$$\frac{\alpha}{1-\alpha} \cdot \frac{x_2}{x_1} = \frac{r_1}{r_2}$$

$$\Rightarrow (1-\alpha) r_1 x_1 = \alpha r_2 x_2$$

$$\Rightarrow (1-\alpha) r_1 x_1 - \alpha r_2 x_2 = 0$$

Prob. 5 If PCC for x is parallel to x axis then derive the shape of the demand curve for x.

Solution: Let $M = P_x X + P_y Y$ be the budget constraint of the consumer; where P_x and P_y are the prices of X and Y and M be the constant money income of the consumer.

Now from total differentiation of the budget constraint we get:

$$dM = P_x\,dx + x\,dP_x + P_y\,dy + y\,dP_y$$

Hence as M is constant $dM = 0$. Again as PCC for x is parallel to x axis, $dy = 0$. Moreover, as we consider the demand curve for x, price of y is assumed to be constant, i.e, $dP_y = 0$. So that we have

$$P_x\,dx + x\,dP_x = 0$$

$$\text{or}\quad x\,dP_x = -P_x\,dx$$

$$\text{or}\quad \frac{dP_x}{P_x} = -\frac{dx}{x}$$

Integrating both sides we get:

$$\int \frac{dP_x}{P_x} = -\int \frac{dx}{x}$$

$$\Rightarrow \log P_x = -\log x + \log c$$

where $\log c$ is integrating constant.

$$\Rightarrow \log(P_x x) = \log C$$

$$\text{or}\quad P_x x = C$$

This is an equation of a rectangular hyperbola. This shows that the shape of the demand curve will be rectangular hyperbola.

Prob. 4 Obtain the Engel curve of a consumer whose utility function is $U = q_1 q_2$.

Solution: We know that at consumer's equilibrium:

$$MRS_{q_1,q_2} = \frac{p_1}{p_2}$$

where p_1 and p_2 are constant unit prices of two commodities Q_1 and Q_2 respectively.

$$\text{Now}\quad MRS_{q_1,q_2} = \frac{MU_1}{MU_2}$$

$$\text{where}\quad MU_1 = \frac{\partial U}{\partial q_1} = q_2$$

$$\text{and}\quad MU_2 = \frac{\partial U}{\partial q_2} = q_1$$

Therefore, at equilibrium

$$\frac{q_2}{q_1} = \frac{p_1}{p_2}$$

$$\text{or}\quad q_2 = \frac{p_1}{p_2} q_1$$

Substituting the value of q_2 into the budget equation we get:

$$p_1 q_1 + p_2 \times \frac{p_1}{p_2} q_1 = M$$

Where 'M' denotes constant money income of the consumer.

Therefore,

$$2p_1 q_1 = M$$

$$\Rightarrow q_1 = \frac{M}{2p_1}$$

This is the equation of Engel curve.

Prob. 6 Utility function for two goods is given by $U = (x+2)(y+1)$. It is given that $P_x = Rs.\ 4$, $P_y = Rs.\ 6$ and the individual's fixed income is Rs. 130. Using the Lagrange multiplier method, find the optimum levels of purchase of the two commodities. Is the second order condition for maximum utility satisfied?

Solution: Here the aim of the consumer is to maximize –

$$U = (x+2)(y+1)$$

Subject to the budget constraint:

$$4x + 6y = 130$$

The Lagrangian function to solve the problem becomes:

$$z = (x+2)(y+1) + \lambda(130 - 4x - 6y)$$

The first order condition for utility maximization require:

$$z_x = \frac{\partial z}{\partial x} = (y+1) - 4\lambda = 0 \quad \text{——} (1)$$

$$z_y = \frac{\partial z}{\partial y} = (x+2) - 6\lambda = 0 \quad \text{——} (2)$$

$$z_\lambda = \frac{\partial z}{\partial \lambda} = 130 - 4x - 6y = 0 \quad \text{——} (3)$$

From (1) we get

$$y + 1 = 4\lambda \quad \text{——} (4)$$

and from (2) we get

$$(x+2) = 6\lambda \quad \text{——} (5)$$

Now (4) ÷ (5) gives

$$\frac{y+1}{x+2} = \frac{4}{6}$$

$$\Rightarrow 4x + 8 = 6y + 6$$

$$\Rightarrow 4x = 6y - 2 \quad \text{——} (6)$$

Substituting (6) into (3) we get

$$130 - (6y - 2) - 6y = 0$$

$$\Rightarrow 12y = 132$$

$$\Rightarrow y = 11 \quad \text{——} (7)$$

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Again substituting (7) into (6) we get

$$4x = 6 \times 11 - 2$$

$$\Rightarrow 4x = 64$$

$$\Rightarrow x = 16$$

Therefore,

$$x = 16 \text{ units}$$

$$\text{and } y = 11 \text{ units}$$

Now for 2nd order condition we have to find:

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = 0, \quad z_{xy} = \frac{\partial^2 z}{\partial x \partial y} = 1, \quad z_{x\lambda} = \frac{\partial^2 z}{\partial x \partial \lambda} = -4$$

Again

$$z_{yx} = \frac{\partial^2 z}{\partial y \partial x} = 1, \quad z_{yy} = \frac{\partial^2 z}{\partial y^2} = 0, \quad z_{y\lambda} = \frac{\partial^2 z}{\partial y \partial \lambda} = -6$$

$$\text{and } z_{\lambda\lambda} = \frac{\partial^2 z}{\partial \lambda^2} = 0$$

Therefore, the value of Bordered-Hessian determinant becomes:

$$\begin{vmatrix} z_{xx} & z_{xy} & z_{x\lambda} \\ z_{yx} & z_{yy} & z_{y\lambda} \\ z_{x\lambda} & z_{y\lambda} & z_{\lambda\lambda} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & -4 \\ 1 & 0 & -6 \\ -4 & -6 & 0 \end{vmatrix} = -1(0 - 24) - 4(-6 - 0) = 48 > 0$$

Since the value of the Bordered Hessian determinant is positive, 2nd order condition for utility maximization is satisfied. So that $x = 16$ units and $y = 11$ units become the optimum purchases of the consumer.

Prob. 7 Let the utility function be given by $U = xy$ and the budget constraint be given as $100 - P_x x - P_y y = 0$. (i) Find the demand functions for x and y. (ii) Show that these functions are homogeneous of degree zero in absolute prices & income.

Solution: We know that the aim of the consumer is to maximize
(i) $U = xy$ subject to the budget constraint $100 - P_x x - P_y y = 0$.

The Lagrangian function to solve the problem becomes:

$$Z = xy + \lambda(100 - P_x x - P_y y)$$ where λ is the Lagrangian multiplier.

The first order conditions for utility maximization require:

$$Z_x = \frac{\partial Z}{\partial x} = y - \lambda P_x = 0 \quad \text{—— (1)}$$

$$Z_y = \frac{\partial Z}{\partial y} = x - \lambda P_y = 0 \quad \text{—— (2)}$$

and $$Z_\lambda = \frac{\partial Z}{\partial \lambda} = 100 - P_x x - P_y y = 0 \quad \text{—— (3)}$$

From (1) we get

$$y = \lambda P_x \quad \text{—— (4)}$$

From (2) we get

$$x = \lambda P_y \quad \text{—— (5)}$$

Now (4) ÷ (5) gives

$$\frac{y}{x} = \frac{P_x}{P_y}$$

$$\therefore y = \frac{P_x}{P_y} x \quad \text{—— (6)}$$

Substituting (6) into (3) we get

$$100 - P_x x - P_y \cdot \frac{P_x}{P_y} x = 0$$

$$\Rightarrow 2 P_x x = 100$$

$$\Rightarrow x = \frac{50}{P_x} \quad \text{—— (7)}$$

Again substituting (7) into (6) we get

$$y = \frac{50}{P_y} \quad \text{—— (8)}$$

Equations (7) and (8) represent respectively the demand functions for X and Y.

P.T.O

provided that the 2nd order condition for utility maximization is satisfied.

(ii) If P_x, P_y and 50 are increased in 'λ' proportion then we have:

$$x = \frac{\lambda 50}{\lambda P_x} = \frac{50}{P_x}$$

and $$y = \frac{\lambda 50}{\lambda P_y} = \frac{50}{P_y}$$

Therefore, we see that even if all the demand determinants are increased by the same proportion quantity demand for X and Y remain unchange. Hence the demand functions are homogeneous of degree zero in prices and income.

Prob. 8 An individual's utility function is given by $U = x^\alpha y^\beta$. If P_x and P_y are the prices of goods X and Y and the individual's income is M, find the demand functions. Show that price and income elasticities for either good is equal to unity in absolute value.

Solution: The aim of the consumer is to maximize

$$U = x^\alpha y^\beta$$

Subject to the budget constraint

$$M = P_x x + P_y y$$

The Lagrangian function to solve the problem becomes:

$$Z = x^\alpha y^\beta + \lambda(M - P_x x - P_y y)$$ where λ is the Lagrangian multiplier.

First order conditions for maximum utility require:

$$Z_x = \frac{\partial Z}{\partial x} = \alpha x^{\alpha-1} y^\beta - \lambda P_x = 0 \quad \text{—— (1)}$$

$$Z_y = \frac{\partial Z}{\partial y} = \beta x^\alpha y^{\beta-1} - \lambda P_y = 0 \quad \text{—— (2)}$$

$$Z_\lambda = \frac{\partial Z}{\partial \lambda} = M - P_x x - P_y y = 0 \quad \text{—— (3)}$$

From (1) we get

$$\alpha x^{\alpha-1} y^\beta = \lambda P_x \quad \text{—— (4)}$$

and from (2) we get

$$\beta x^\alpha y^{\beta-1} = \lambda P_y \quad \text{—— (5)}$$

P.T.O

Now ④ ÷ ⑤ gives

$$\frac{\alpha x^{\alpha-1} y^{\beta}}{\beta x^{\alpha} y^{\beta-1}} = \frac{\lambda P_x}{\lambda P_y}$$

$$\Rightarrow \frac{\alpha}{\beta} \cdot \frac{y}{x} = \frac{P_x}{P_y}$$

$$\Rightarrow y = \frac{\beta}{\alpha} \cdot \frac{P_x}{P_y} x \quad \text{⑥}$$

Substituting ⑥ into ③ we have:

$$M - P_x x - P_y \cdot \frac{\beta}{\alpha} \cdot \frac{P_x}{P_y} x = 0$$

$$\Rightarrow P_x x + \frac{\beta}{\alpha} P_x x = M$$

$$\Rightarrow \left(\frac{\alpha+\beta}{\alpha}\right) P_x x = M$$

$$\Rightarrow x = \frac{\alpha}{\alpha+\beta} \cdot \frac{M}{P_x}$$

This is the demand function for X.

Now substituting 'x' into equation ⑥ we have

$$y = \frac{\beta}{\alpha} \cdot \frac{P_x}{P_y} \cdot \frac{\alpha}{\alpha+\beta} \cdot \frac{M}{P_x}$$

$$\Rightarrow y = \frac{\beta}{\alpha+\beta} \cdot \frac{M}{P_y}$$

This is the demand function for Y.

Hence $\frac{\partial x}{\partial P_x} = -\frac{\alpha}{\alpha+\beta} \cdot \frac{M}{P_x^2}$

Therefore, absolute value of the price elasticity of demand for x:

$$|e_{P_x}| = -\frac{P_x}{x} \cdot \frac{\partial x}{\partial P_x} = -\left(\frac{P_x}{x}\right)\left(-\frac{\alpha}{\alpha+\beta} \cdot \frac{M}{P_x^2}\right)$$

Substituting 'x' we have

$$|e_{P_x}| = -P_x \frac{P_x(\alpha+\beta)}{\alpha M} \cdot \left(-\frac{\alpha}{\alpha+\beta} \cdot \frac{M}{P_x^2}\right)$$

$$\Rightarrow |e_{P_x}| = 1$$

So that absolute value of the price elasticity of demand is unity.

Again $\frac{\partial x}{\partial M} = \frac{\alpha}{(\alpha+\beta)P_x}$

Therefore, income elasticity of demand for x:

$$\eta_x = \frac{M}{x} \cdot \frac{\partial x}{\partial M} = \frac{M}{x} \cdot \frac{\alpha}{(\alpha+\beta)P_x}$$

Now substituting $x = \frac{\alpha}{\alpha+\beta} \cdot \frac{M}{P_x}$ we have

$$\eta_x = \frac{M(\alpha+\beta)P_x}{\alpha M} \times \frac{\alpha}{(\alpha+\beta)P_x} = 1$$

This shows that income elasticity of demand for good x is unity too.

Prob. 9 Prove that price elasticity of demand

$$|e_p| = \frac{AR}{AR - MR}$$

and hence verify the relation for the linear demand law $p = a - bq$ where $a > 0$, $b > 0$

Solution: Suppose at price 'p' per unit 'q' units of output are sold. Then total revenue $(TR) = pq$. Therefore, average revenue $(AR) = \frac{TR}{q} = p$ and marginal revenue $(MR) = p + q\frac{dp}{dq} = p\left(1 + \frac{q}{p}\frac{dp}{dq}\right)$

$$\Rightarrow MR = p\left(1 - \frac{1}{-\frac{p}{q}\frac{dq}{dp}}\right)$$

$$\Rightarrow MR = p\left(1 - \frac{1}{|e_p|}\right)$$

$$\Rightarrow MR = AR\left(1 - \frac{1}{|e_p|}\right) \text{ where } p = AR$$

$$\Rightarrow MR = AR - \frac{AR}{|e_p|}$$

$$\Rightarrow \frac{AR}{|e_p|} = AR - MR$$

$$\Rightarrow |e_p| = \frac{AR}{AR - MR}$$

Now for the linear demand law

$$p = a - bq$$

we have

$$TR = pq = aq - bq^2$$

$$\therefore \quad MR = \frac{d(TR)}{dq} = a - 2bq$$

Again $AR = a - bq$ as $p = AR$

$$\therefore \frac{AR}{AR - MR} = \frac{p}{a - bq - a + 2bq} = \frac{p}{bq} = \frac{p}{a-p} \quad \text{as } p = a - bq$$

Further

$$\frac{dp}{dq} = -b$$

$$\therefore \quad \frac{dq}{dp} = -\frac{1}{b}$$

Therefore,

$$|e_p| = -\frac{p}{q} \cdot \frac{dq}{dp} = -\frac{p}{q} \times -\frac{1}{b} = \frac{p}{bq} = \frac{p}{a-p}$$

as $p = a - bq$

So for the linear demand law $p = a - bq$

$$|e_p| = \frac{AR}{AR - MR} \text{ is proved.}$$

Prob. 10 A monopolist faces the demand function $p = a - q$, where $a > 0$. His marginal cost of production is constant 'c'. A tax per unit of production is imposed. Find the value of tax rate (t) that will maximize the tax revenue. What are the values of post tax price and quantity?

Solution: Given that marginal cost $(MC) = \frac{dC}{dq} = c$ (constant).

This implies

$$dC = c\,dq$$

Integrating both sides we get

$$\int dC = c \int dq$$

$$\text{or} \quad C = cq + F$$

where F is integrating constant. 'F' denotes total fixed cost. Therefore, total cost function becomes

$$C = cq + F$$

Now if a tax at the rate 't' per unit of production is imposed then the total cost function becomes

$$C = cq + F + tq$$

Thus after imposition of tax profit(π) of the monopolist becomes:

$$\pi = pq - C = (a-q)q - cq - F - tq$$

$$\Rightarrow \pi = aq - q^2 - cq - F - tq$$

Now profit of the monopolist will be maximum when

$$\frac{d\pi}{dq} = a - 2q - c - t = 0$$

$$\Rightarrow 2q = a - c - t$$

$$\Rightarrow q = \frac{a - c - t}{2}$$

This is the post-tax profit maximizing level of output.

Therefore, post-tax price becomes:

$$P = a - \frac{a-c-t}{2} \quad [\text{Substituting the value of } q]$$

$$= \frac{2a-a+c+t}{2}$$

$$= \frac{a+c+t}{2}$$

Now at equilibrium output level total tax revenue (T) becomes:

$$T = tq = t\left(\frac{a-c-t}{2}\right) = \frac{at-ct-t^2}{2}$$

This tax revenue, with respect to tax rate 't', will be maximized when

$$\frac{dT}{dt} = 0$$

i.e, when $\frac{a-c-2t}{2} = 0$

$$\Rightarrow 2t = a-c$$

$$\Rightarrow t = \frac{a-c}{2}$$

This is the tax rate at which tax revenue of the monopo will be maximized.