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End Semester Examination of Semester–III, 2015
Subject : MATHEMATICS (HONS.) (UG)
Paper : VI (Theory)
Full Marks : 40
Time : 2 Hrs

The figures in the margin indicate the marks corresponding to the question.

Candidates are requested to give their answers in their own word as far as practicable.

Illustrate the answers wherever necessary.

Group A

Answer **any two** out of four questions : 10x2=20

1. a) Let A be a subset of metric space X. Prove that A is open if and only if Int A = A. 4

   b) Consider $\mathbb{Q}$, the set of all rational number, as a metric space, with $d(p, q) = |p - q|$; $p, q \in \mathbb{Q}$. Let $E = \{p \in \mathbb{Q} : 2 < p^2 < 3\}$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. 6

2. a) Let $f : S \rightarrow S$ $(S \subset R)$ be continuous on S and S be a compact set. Then prove that f(s) is a Compact Set and also f attains its bounds. 4+1

   b) An open set in R is the union of a Countable family of disjoint open intervals. Is the converse true if yes state only why it is true. 4+1

3. a) Let f be a mapping from a metric space X into a metric space Y. Show that f is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y. 5

b) Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable.

Define $F:[a,b]\to\mathbb{R}$ by F(a) = 0 and

$F(x)=\int_a^x f(t)dt,\ x\in(a,b]$. Show that F is uniformly continuous. 5

4. a) If f : [a, b]→R be piecewise continuous on [a, b] then prove that f is Riemann integrable on [a, b]. 5

b) Let $\mathbb{R}[a,b]$ be the set of all Riemann integrable functions on [a, b]. If $f\in\mathbb{R}[a,b]$,

show that $f^2\in\mathbb{R}[a,b]$.

Give an example of a bounded function f on [a, b] for which $f^2\in\mathbb{R}[a,b]$, but $f\notin\mathbb{R}[a,b]$. 5

Group B

Answer **any two** out of four questions : 6X2=12

5. a) Let A, B be disjoint closed subsets of $\mathbb{R}$.
Prove that $f:\mathbb{R}\to\mathbb{R}$ given by

$$f(x)=\frac{d(x,A)}{d(x,A)+d(x,B)}$$

is continuous function on $\mathbb{R}$ satisfying $0 \le f(x) \le 1$ for all $x \in \mathbb{R}$ and

$$f(x) = 0, \; x \in A$$
$$= 1, \; x \in B$$

6. a) Find the asymptotes of the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ where $h^2 > ab$. 3

b) Examine the Convergence of $\int_0^1 \frac{x^{p-1}}{1-x} dx$. 3

7. a) Let $f:(a,b) \to \mathbb{R}$ be differentiable. Let $a < x_n < c < y_n < b$ be such that $y_n - x_n$ tends to 0 as $n \to \infty$.

Show that $\lim_{x \to \infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(c)$. 4

b) Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable at $x = 0$.
Define $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x^2)$ for all $x \in \mathbb{R}$.
Show that g is differentiable at $x = 0$. 2

8. a) Prove that for equiangular spiral $r = a e^{\theta \cot \alpha}$ the radius of curvature subtends a right angle at the pole. 3

b) If $f:[a,b] \to \mathbb{R}$ be continuous function and f(x) always a rational number then f(x) is a constant. 3

Group C

Answer **any four** out of eight questions : 4x2=8

9. Write down explicitly the expansion for the nth derivative of the function $f(x) = x^2e^{3x}$.

10. Use Taylor theorem to prove that

$$\cos x > 1 - \frac{x^2}{2} \text{ for } -\pi < x < \pi.$$

11. Prove that an enumerable subset of R is a set of measure zero.

12. Show that the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_0, y_0) on it has the equation $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$.

13. Find the radius of curvature at the origin of the curve $y^2 - 3xy + 2x^2 - x^3 + y^4 = 0$.

14. The curve $y=\sqrt{1+x^2}\sin\left(\frac{1}{x}\right)$. What are its asymplotes?

15. Let $f: I \to \mathbb{R}$ be a Lipschitz function on I. Then prove that f is uniformly Continuous on I.

16. Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If $\lim_{x\to 0} f(x)$ exists, show that $\lim_{x\to p} f(x)$ exists for each $p\in\mathbb{R}$.