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End Semester Examination of Semester–II, 2016
Subject : MATHEMATICS (PG)
Paper : MTMPG-203
Full Marks : 40
Time : 2 Hrs

The figures in the margin indicate the marks corresponding to the question

Candidates are requested to give their answers in their own word as far as practicable.

Illustrate the answers wherever necessary

Group A

Answer **any one** out of two questions : 10X1=10

1. a) Let $P_0(x)$, $P_1(x)$, $P_2(x)$, ..., $P_n(x)$ be continuous functions of x for $a \leq x \leq b$ and $y_1(x)$, $y_2(x)$, ... , $y_n(x)$ be n solutions of equation

$$L[y(x)] \equiv P_0 \frac{d^n y}{dx^n} + P_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots\ldots\ldots + P_{n-1}(x) \frac{dy}{dx} + P_n(x) y = 0$$

Then prove that n solutions are linearly independently if and only if $W(y_1, y_2, \ldots\ldots, y_n)_{x=x_0} \neq 0$ at least one point x_0 is $a \leq x \leq b$ where $W(y_1, y_2, \ldots\ldots, y_n)_{x=x_0}$ denotes the Wrongskian of y_1, y_2, ..., ..., y_n at $x = x_0$.

b) Prove that $(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x)$ where $P_n(x)$ denotes Legendre polynomial of degree n. 4

2. a) Derive the series solution of the differential equation

$$x(1-x)\frac{d^2y}{dx^2}-3x\frac{dy}{dx}-y=0 \text{ near } x=0.$$ 7

b) Prove that $\cos x = J_0(x)+2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$, where $J_n(x)$ is the Bessel function. 3

Group B

Answer **any one** out of two questions : 6X1=6

3. Using Green's method solve the equation $\frac{d^2u}{dx^2}=\sin x,\ 0\le x\le 1$ subject to the boundary condions $U(0) = \alpha$, $u'(1) = \beta$. 6

4. Find the general solution of

$$(x^2+1)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=6(x^2+1)^2$$

given that $y = x$ and $y = x^2 - 1$ are linearly independent solutions of the corresponding homogenous equation.

Group C

Answer **any two** out of four questions : 2X2=4

5. Show that $x = \alpha$ is not a regular singular point of the

equation $x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+\left(x^2-n^2\right)u=0$

where n is a parameter.

6. Show that the boundary value problem,

$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+\lambda u=0$, u(0) = 0, u(π) = 0, is a Strum-Lionville problem.

7. Prove that the functions φ_1, φ_2 and φ_2 defined by

$\varphi_1(x)=\begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix}$ $\varphi_2(x)=\begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix}$ and $\varphi_3(x)=\begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix}$

are fundamental solution of the equation

$$\frac{dx}{dt}=\begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x \text{ where } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

8. Show that the function $f(t,x)=\left(x+x^2\right)\frac{\cos t}{t^2}$ satisfies Lipschitz condition in $|x| \leq 1$ and $|t-1|<\frac{1}{2}$ and find the Lipschitz constant.

Group D

Answer **any one** out of two questions : 10X1=10

9. a) Define Saddle-node bifurcation of a dynamical system. Show that the system $\dot{x}=r-x-e^{-x}$ undergoes a Saddle-node bifurcation as r is a varied and find the value of r at the bifurcation point. 5

b) Plot the phase portrait and classify the fixed point of the following linear system. 5

$$\dot{x}=-3x+4y$$

$$\dot{y}=-2x+3y$$

10. a) Consider the system

$$\dot{x}=-y+ax\left(x^2+y^2\right)$$

$$\dot{y}=x+ay\left(x^2+y^2\right)$$

Where a is a parameter. Show that the linearized system incorrectly predicts that the origin is a center for all values of a, whereas in fact the origin is a stable spiral if $a < 0$ and an unstable spiral if $a > 0$. 5

b) Show that when $a > 0$ and $b > 0$ all solutions of $\ddot{x}+a\dot{x}+\left(b+ce^{-t}\cos t\right)x=0$ are asymptotically stable for $t \geq t_0$ for any y_0. 5

Group E

Answer **any one** out of two questions : 6x1=6

11. Consider a particle of mass m = 1 moving in a double-well potential $V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$. Find and classify all the equilibrium points for the system. Then plot the phase portrait of the system.

12. State and prove Floquet's theorem for linear ODE with periodic co-efficients.

Group F

Answer **any two** out of four questions : 2x2=4

13. Discuss the stability of the point (0, 0) of the system

$$\frac{dx}{dt} = -x - y - x^3$$

$$\frac{dy}{dt} = x - y - y^3$$

Using suitable Liapunov function.

14. Draw the phase diagram of the system.

$$\frac{dx}{dt} = x + 7y$$

$$\frac{dy}{dt} = 3x + 5y$$

15. Examine the critical points of the non-linear plane autonomous system.

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = x^2 - 4x + \lambda$$

Where λ is a parameter.

16. Show that the equation with periodic co-efficient but has no periodic solution.

$$\dot{x} = P(t)x$$

Where $x = \begin{pmatrix} x \\ y \end{pmatrix}$ and $p(t) = \begin{pmatrix} 1 & \cos t \\ 0 & -1 \end{pmatrix}$