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End Semester Examination of Semester–III, 2015
Subject : MATHEMATICS (PG)
Paper : MTM–306 (IIA, Special paper)
(Theory of Relativity)
Full Marks : 40
Time : 2 Hrs

The figures in the margin indicate the marks corresponding to the question.

Candidates are requested to give their answers in their own word as far as practicable.

Illustrate the answers wherever necessary.

Group A

Answer **any two** out of four questions : 10x2=20

1. a) Define inertial frame. Write down the postulates of special theory of Relativity. 2+2

   b) Derive the Lorentz transformation that connects two inertial frames. 6

2. a) Let M be an n-dimensional manifold. Let $p \in M$. Show that $\dim T_pM = n$, where T_pM is the tangent space at p. 6

   b) Define commutator $[x, y]$ of two smooth vector fields X, Y on a manifold M. Show that the smooth vector fields X, Y, Z on a manifold M satisfies the identity

   $$[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0.$$ 1+3

3. a) Define connection on a smooth manifold. 3

   b) Show that on a semi-Riemannian manifold M there is a unique connection ∇ such that $[V, W] = \nabla_V W - \nabla_W V$

   and $X\langle V, W\rangle = \langle \nabla_X V, W\rangle + \langle V, \nabla_X W\rangle$

   for all $X, V, W \in \mathfrak{X}(M)$. 7

4. a) Consider the Schwarzschild metric given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

   Describe the geometry in the region $r < 2m$ using Kruskal-Szekeres coordinates. 8

   b) Write down the Einstein's field equation in General relativity. 2

Group B

Answer **any two** out of four questions : 6X2=12

5. Let in a Semi-Riemannian manifold all the components $g_{\mu\nu}$ of the metric tensor are independent of the co-ordinate x^0.

   Show that $g_0\mu \frac{dx^\mu}{d\tau}$ is constant along the geodesic trajectory.

6. a) Define tangent bundle on a differentiable manifold. 1

b) Define vector field on a differentiable manifold. What do you mean by an integral curve on a vector field? 2

c) If V is a vector field on a differentiable manifold M, then show that for each $p \in M$ there is an interval $I \subseteq \mathbb{R}$ around 0 and a unique integral curve $\gamma : I \to M$ of V such that $\gamma(0) = p$. 3

7. Define exponential map of a differential manifold M at some point of the manifold. Show that for each point $p \in M$ there exists a neighbourhood V of 0 in T_pM on which the exponential map is a diffcomosphism on to a neighbourhood U of p in M. 2+4

8. Define Minkowski spacetime. Write down the Mnikowski metric in spherical polar co-ordinates. What do you mean by a Cauchy surface in a spacetime? Identify Cauchy surfaces in a Minkowski spacetime. 2+1+2+1

Group C

Answer **any four** out of eight questions : 2X4=8

9. What is a pull back map between two differentiable manifolds?

10. Define Kronceker detla independently of bases.

11. State Birkoff's theorem regarding Schwardzschild solution of Einstein's field equations.

12. State the equivalence principle regarding General relativity.

13. Write down the Bianchi identity for Riemann curvature tensor.

14. What do you mean by a geodesically incomplete manifold?

15. What is a geodesic in a manifold?

16. Consider the (0, 2) symmetric tensor S on a two-dimensional manifold, whose components in a co-ordinate system (x, y) are given by

$$S_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix}$$

Determine the components of the tensor in the co-ordinates system (x', y') given by

$$x' = \frac{2x}{y}, \quad y' = \frac{y}{2}$$