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End Semester Examination of Semester–III, 2015

Subject : MATHEMATICS (PG)

Paper : MTM–303 (Theory)

Full Marks : 40

Time : 2 Hrs

The figures in the margin indicate the marks corresponding to the question.

Candidates are requested to give their answers in their own word as far as practicable.

Illustrate the answers wherever necessary.

Group A

Answer **any two** out of four questions : 10X2=20

1. a) Show that the Green's function for the equation

$$\frac{\partial^2 u}{\partial x \partial y}+u=0 \text{ is } v(x,y;\, z,\eta)=J_0\sqrt{2(x-z)(y-\eta)}$$

where J_0 denoted Bessel's function of the first kind of order zero. 5

b) Prove that $\lim\limits_{\varepsilon\to 0+}\frac{1}{x+i\varepsilon}=-\pi i\,\delta(x)+\wp\frac{1}{x}$. 5

2. a) Solve the following equation

$$\left(D^2+2DD'+D'^2-2D-2D'\right)u=\sin(x+2y).$$

Where $D\equiv\frac{\partial}{\partial x}$ and $D'\equiv\frac{\partial}{\partial y}$. 5

b) Show that the Gaussian Sequence $\{F_\varepsilon(z)\}$,

where $f_\varepsilon(x) = \frac{1}{2\sqrt{\pi\varepsilon}} e^{-\frac{x^2}{4\varepsilon}}$ from the theory of statistics,

converges to a delta function as $\varepsilon \to 0+$. 5

3. a) Use Fourier transform method to solve the PDE

$$\frac{\partial^2 u}{\partial t^2} = e^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty.$$

and ICs : $u(x,0) = f(x), \quad -\infty < x < \infty.$

$u_t(x,0) = 0$

b) Solve by separation of variable method

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad \alpha > 0.$$ 5+5

4. a) Consider the case when $\mathbb{R}$ consists of the half-plane defined by $x \geq 0, \; -\infty < y < \infty$ and hence solve $\nabla^2 u = 0$ in the region subject to the condition $u = f(y)$ on $x = 0$, using the Green's function technique.

b) Find the characteristics of the equation $pq = z$ and hence determine the integral surface which passes through the parabola $x = 0, \; y^2 = z$. 5+5

Group B

Answer **any two** out of four questions : 6x2=12

5. Find the integral surface of the linear PDE $x(y^2 + z)\, p - y(x^2 + z)q = (x^2 - y^2)z$. containing the straight line $x + y = 0,\ z = 1$. 5

6. Let u be harmonic in a region $\mathbb{R}$. Also, let P(x, y, z) be a given posit in $\mathbb{R}$ and S (P; r) be a sphere with centre at P such that S(p; r) is completely contained in the domain of harmonicity of u. Then prove that

$$u(P)=\bar{u}(r)=\frac{1}{4\pi r^2}\iint_{S(P;r)} u(Q)dS.$$ 5

7. Find the solution of the one-dimensional diffusion equation satisfying the following BCs

   i) T is bounded as $t \to \infty$.

   ii) $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \quad \forall t$

   iii) $\left.\frac{\partial T}{\partial x}\right|_{x=a} = 0 \quad \forall t$

   and iv) $T(x, 0) = x\,(a - x), \quad 0 < x < 9.$

8. Show that the equality $f_\alpha * f_\beta = f_{\alpha+\beta}$

   a) Where $f_\alpha(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1},\ \alpha>0.$

b) Prove that $\left\langle P\left\{H(x)x^{\lambda-K}\right\}, \phi\right\rangle$

$$=\int_0^{\infty} x^{\lambda-K}\left[\phi(x)-\sum_{m=0}^{K}\frac{\phi^{(m)}(0)}{m!}x^m\right]dx,$$

$-1 < \lambda < 0$, $K = 1, 2, ...$ 2+3

Group C

Answer **any four** out of eight questions : 2x4=8

9. Prove that the Laplace operator is a self-adjoint operator.

10. If a harmonic function vanishes everywhere on the boundary, then it is identically zero everywhere.

11. Find the characteristic curves of the equation

$$x^2u_{xx}-y^2u_{yy}=x^2y^2+x,\ x>0.$$

12. Evaluate : $\int_0^{\pi}\cosh x\,\delta''(x-1)\,dx$.

13. Classify and reduce the relation $y^2u_{xx} - 2xyu_{xy} + x^2u_{yy}$ $= \frac{y^2}{x}u_x+\frac{x^2}{y}u_y$ to a canonical form.

14. Show that the equation $u_{xx} + \frac{2N}{x} u_x = \frac{1}{a^2} u_{tt}$ where N and a are constants, is a hyperbolic form.

15. Explain the characteristic triangle of D'Alembert's one-dimensional wave equation.

16. Prove that the delta function is a singular distribution.