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End Semester Examination of Semester–III, 2015

Subject : MATHEMATICS (PG)

Paper : MTM–301 (Theory)

(Functional Analysis)

Full Marks : 40

Time : 2 Hrs

The figures in the margin indicate the marks corresponding to the question.

Candidates are requested to give their answers in their own word as far as practicable.

Illustrate the answers wherever necessary.

Group A

Answer **any two** out of four questions : 10x2=20

1. a) What do you mean by a separable metric space? Show that every compact metric space is separable. 1+5

   b) Show that the closed ball $\overline{B}=\{x\in C[0,1]:\sup|x(t)\le 1|\}$ of C[0, 1] with submetric is not compact. 4

2. a) What do you mean by an equicontinuous subset of C(X), where X is a compact metric space? Show that a subset M of C[a, b] is compact if M is closed, bounded and equicontinous. 1+6

b) Let (X, ||, ||) be a NLS and T : X→X be a bounded linear operator. Let

$\|T\| = \text{Inf}\{M > 0 : \|T_x\| \leq M\|x\| \text{ for all } x \in X\}$

Show that

$\|T\| = \sup\{\|T_x\| : \|x\| \leq 1\}$. 3

3. a) Prove that every inner product space is a normed linear space (NLS). 3

b) Let $\{x_i\}_{i \in I}$ be an orthonormal family of vectors in a inner product space X. Show that

$$\sum_{i \in I} |\langle x, x_i \rangle|^2 \leq \|x\|^2$$

holds for each $x \in X$. 5

c) If in an inner product space, $\langle x, u \rangle = \langle x, v \rangle$ for all x in the space, show that u = v. 2

4. a) Let X and Y be two Banach spaes, and let T : X→Y be an onto continuous linear operator. Show that if zero is an interior point of a subset A of X, then zero is also an interior point of T(A). 6

b) Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonotmal set in a Hilbert space H where $x \in \mathbb{N}$ is fixed. If $x \in H$ be fixed, show that for scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$

$\left\| x - \sum_{i=1}^{n} \alpha_i e_i \right\|$ is minimum, when $\alpha_i = \langle x, e_i \rangle$, i = 1, 2, ..., n. 4

Group B

Answer **any two** out of four questions : 6X2=12

5. Prove that in a NLS the closure of the open unit ball is the closed unit ball. Is the result true in any metric space? Justify your answer. 4+2

6. a) Show that every operator on a finite dimensional NLS is continuous.

   b) If two non-zero linear functions f_1, f_2 over a linear space have the same null space, then show that f_1 and f_2 are proportional. 4+2

7. Let A be a non-empty closed convex subset of a Hilbert space H. Show that for each $x \in H$, there exists unique $y \in A$ such that $d(x, A) = \|x - y\|$. 6

8. a) Let M be a closed subspace of a Hilbert space H. Show that $H = M \oplus M^{\perp}$. 4

   b) Consider $E = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{R})$ Define

   $T: M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by $T(A) = EA$ for $A \in M_2(\mathbb{R})$.

   Show that T is a linear operator. 4+2

Group C

Answer **any four** out of eight questions : 2x4=8

9. Let A and B be two subsets of a NLS X. Show that $A + B = \{a + b : a \in A, b \in B\}$ is open if A is open.

10. If C is a convex subset of a NLS X and $x_0 \in X$, then show that $x_0 + C$ is a convex set.

11. State open mapping theorem.

12. If X and Y are Banach spaces, show that the null space N(T) of a closed linear operator $T : X \to Y$ is a closed subspace of X.

13. Show that $\langle z_1, z_2 \rangle = z_1 . \overline{z_2}$, $z_1, z_2 \in \mathbb{C}$ defines an Inner product on $\mathbb{C}$.

14. Define sell adjoint operator on a Hilbert space.

15. Show that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ does not have any eigen vector.

16. If T is a self-adjoint operator on a Hilbert space H, show that T^3 is a self-adjoint operator.