Total Pages : 4

End Semester Examination of Semester–I, 2015

Subject : MATHEMATICS (PG)

Paper : MTM-102 (Linear Algebra)

Full Marks : 40

Time : 2 Hrs

The figures in the margin indicate the marks corresponding to the question

Candidates are requested to give their answers in their own word as far as practicable.

Illustrate the answers wherever necessary

Group A

Answer **any two** questions : 10x2=20

1. a) Let $T : U \to V$ be a linear transformation. Show that T is nonsingular if and only if it takes linearly independent sets into linearly independent sets. 6

   b) Let E be a projection operator on V. Show that if $E \neq I$, then E is singular. 4

2. a) Let U and V be vector spaces over a field F. Show that L (U, V) is isomorphic to $M_{m \times n}(F)$ where $m = \dim V$ and $n = \dim U$. 7

   b) Let V be a vector space over F of dimension n. Show that any $T \in L(V)$ satisfies some non-trivial polynomial $g \in F[x]$ of degree at most n^2. 3

3. a) Find all possible Jordan forms of a matrix A with characteristics and minimal polynomials given by $\Delta_A(x) = (x-2)^4 (x-3)^2$ and $m(x) = (x-2)^2 (x-3)^2$. 6

b) Consider the operator $T \in L(\mathbb{R}^3)$ given by $T(x, y, z) = (9x + y, 9y, 7z)$. Is T diagonalizable? Justify your answer. 4

4. a) Let $\|.\|$ be a norm on $\mathbb{R}^n$. For a matrix $A \in M_n(\mathbb{R})$,

define $\|A\|_M = \sup\limits_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}$

Show that $\|.\|_M$ is a norm on $M_n(\mathbb{R})$. 5

b) Let $x_1, x_2, \ldots, x_k$ form an orthonormal set in a real inner product space. Show that

$$\left\|\sum_{i=1}^{K} \alpha_i x_i\right\|^2 = \sum_{i=1}^{K} |\alpha_i|^2$$

Where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are real scalars. 5

Group B

Answer **any two** questions : 6x2=12

5. Consider the operators $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$. Show that T is invertible. Find a formula for T^{-1}. 3+3

6. Let V be a finite dimensional vector space. Let $T \in L(V)$ and the characteristics polynomial $\Delta_T(x)$ be a product of linear factors. Prove that T is diagonalizable. 6

7. a) What do you mean by minimal polynomial of a linear operator on a finite dimensional vector space V? 2

   b) Let $T \in L(V)$ and λ be an eigen value of T. Let $V_\lambda =$ $\{v \in V : v \text{ is an eigen vector corresponding to } \lambda\}$. Show that $V_\lambda = \text{Ker}(\lambda I - T)$. 4

8. a) Show that $\langle A, B \rangle = \text{tr}(B^*A)$ is an inner product on $M_n(\mathbb{C})$. 3

   b) Find an orthonormal basis of the subspace of $\mathbb{R}^4$ spanned by $(2, -1, 0, 1)$, $(6, 1, 4, -5)$ and $(4, 1, 3, -4)$ 3

Group C

Answer **any four** questions : 4X2=8

9. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined by $T(x, y, z) = (x + y, 2z - x)$. Find the matrix of T relative to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$.

10. Let $\{\ell_i\}$ be the standard basis for $\mathbb{R}^3$, and consider the basis $f_1 = (1, 1, 1)$, $f_2 = (1, 1, 0)$ and $f_3 = (1, 0, 0)$. Find the transition matrix P from $\{\ell_i\}$ to $\{f_i\}$.

11. Find all possible norms on $\mathbb{R}^1$.

12. When is $\|x\| = \sum_{j=1}^{n} \alpha_j |x_j|$ a norm on $\mathbb{R}^n$? Justify your answer.

13. Show that an inner product space is a metric space.

14. Suppose $T \in L(V)$ and dim $V = n$. Suppose T has n linearly independent eigen-vectors. What can you say about the matrix representation of T?

15. If S, T be two linear operators on the vector space V(F) such that $ST - TS = I$, then show that $ST^2 - T^2S = 2T$.

16. Show that the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.