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End Semester Examination of Semester–I, 2015
Subject : MATHEMATICS (PG)
Paper : MTM–101 (Real Analysis)
Full Marks : 40
Time : 2 Hrs

The figures in the margin indicate the marks corresponding to the question

Candidates are requested to give their answers in their own word as far as practicable.

Illustrate the answers wherever necessary.

Group A

(Answer **any two** questions) : 10x2=20

1. a) Show that a function $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation if and only if f can be written as the difference of two increasing functions. 6

   b) If $\int_a^b f\,d\alpha = 0$ for every $f \in c[a,b]$, show that α is constant. 4

2. a) Define $f : [0, 1] \rightarrow [0, \infty]$ by $f(x) = 0$ if x is rational and $f(x) = 2^n$ if x is irrational with exactly $n = 0, 1, 2, \ldots$

leading zeroes in its decimal expansion. Show that f is measurable, and find $\int_0^1 f$. 6

b) Let $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be given by $f(A) = A^2$. Is f differentiable on $M_n(\mathbb{R})$? Justify. 4

3. a) Let A be an invertible operator on $\mathbb{R}^n$. Let B be another operator on $\mathbb{R}^n$ such that $\|B-A\|<\frac{1}{\|A^{-1}\|}$. Show that B is also invertible. 5

b) Let Ω be an open subset of $\mathcal{L}(\mathbb{R}^n)$. Show that the mapping $\phi:\Omega\to\Omega$ defined by $\phi(A)=A^{-1}$ is continous. 5

4. a) Let $f : (a, b) \to \mathbb{R}^n$ be a differentiable mapping. Show that $\|f(x)-f(y)\|\leq|y-x| \sup\limits_{0\leq t\leq 1}\|f'(x+t(y-x))$ For all x, y $\in$ (a, b) 6

b) Let $f:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ be given by

$f(x)=\sqrt{\langle x,x\rangle}=\|x\|$

Where $\langle .,.\rangle$ is usual inner product on $\mathbb{R}^n$..

Show that f is differentiable at each $x \in \mathbb{R}^n \setminus \{0\}$ and the derivative is given by

$$Df(x)h = \frac{\langle x,h \rangle}{\|x\|}$$ 4

Group B

Answer **any two** questions : 6x2=12

5. Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be a mapping. What do you mean by f to be C^1? Show that f has continuous partial derivatives if it is C^1. 2+4

6. a) What do you mean by a Lebesque measurable function? 2

   b) Let $f : [a, b] \to \mathbb{R}$ be of bounded variation. Show that f is Lebesgue measurable. 4

7. Let K be a compact subset of $\mathbb{R}^n$, and $\{V_\alpha\}$ be an open cover of K. Show that there exist functions

   $\psi_1, \ldots\ldots \psi_s \in C(\mathbb{R}^n)$ such that

   a) $0 \le \Psi_i \le 1;\ 1 \le i \le s$

   b) each Ψ_i has its support in some V_α, and

   c) $\Psi_1(x) + \ldots\ldots + \Psi_s(x) = 1$ for each $x \in K$. 6

8. a) What do you mean by a Jordan-measurable subset of $\mathbb{R}^2$? 2

b) Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational and } y \text{ irrational} \\ 1 - \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is lowest terms and } y \text{ is rational} \end{cases}.$$

Show that f is integrable and $\int_{[0,1]\times[0,1]} f = 1$. 4

Group C

Answer **any four** questions 4X2=8

9. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that f is differentiable at each $x \in \mathbb{R}^n$ and $Df(x)(h) = f(h),\ h \in \mathbb{R}^n$. 2

10. Evaluate the RS-integral $\int_1^4 2x \, dx^2$. 2

11. State Implicit function theorem. 2

12. Let for $x=(x_1,\dots\dots\dots,x_n)\in\mathbb{R}^n$, $\|x\|_1=\sum_{i=1}^{n}|x_i|$ and

$\|x\|_2=\left(\sum_{i=1}^{n}x_i^2\right)^{\frac{1}{2}}$ show that $\|.\|_1$ and $\|.\|_2$ are equivalent. 2

13. Show that any normal linear space is a metric space. 2

14. Let $(X,\|.\|)$ be a normed linear space. Show that the function $\phi:X\to\mathbb{R}$ given by $\phi(x)=\|x\|$; $x\in X$ is continuous. 2

15. Let X be compact and f : X → Y be continuous. Prove that f is a closed map. 2

16. Let $f:\mathbb{R}^2\to\mathbb{R}$ be given by

$f(x) = 0$ if $x = 0$

$=\dfrac{u^3}{u^2+v^2}$ if $x=(u,v)\neq(0,0)$

Show that the directional derivatives exist at the origin but f is not differentiable at the origin. 2