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End Semester Examination of Semester–I, 2014

Subject : MATHEMATICS (PG)

Paper : 102 (Linear Algebra) (Theory)

Full Marks : 40

Time : 2 Hrs

The figures in the margin indicate the marks corresponding to the question

Candidates are requested to give their answers in their own word as far as practicable.

Illustrate the answers whenever necessary

Group A

(Answer **any two** questions) : 10x2=20

1. Prove that a linear mapping T : V → W is invertible if and only if T is one-to-one and onto, where v and w be vector spaces over a field F. Utilize this result prove that the the linear mapping T : $R^3 \to R^3$ defined by T(x, y, z) = (x + y, y + z, z + x), $(x, y, z) \in \mathbb{R}^3$ is one-to-one and onto. 6+4=10

2. a) Let $M=\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix} \in M_2(\mathbb{R})$. Consider the linear operator $T \in \mathcal{L}(M_2(\mathbb{R}))$ given by TA = AM – MA. Find the basis and dimension of Ker T. 5

b) Let $T \in \mathcal{L}(\mathbb{R}^3)$ and the matrix of T with respect to standard ordered basis is $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix}$

Find a basis for the range of T and a basis for Ker T. 5

3. a) Prove that every finite-dimensional, non-zero, complex vector space has an eigen value. 5

b) Suppose that $S, T \in \mathcal{L}(\mathbb{R}^3)$ such that ST = TS. Prove that null $(T - \lambda I)$ is invariant under for every $\lambda \in \mathbb{R}$. 5

4. a) Let V be an inner product space over a field F. Let $u, v \in V$. Prove that $\langle u, v \rangle = 0$ if and only if

$\|u\| \le \|u + av\|$

for all $a \in F$. 5

b) Let T be self-adjoint operator on a real inner-product space V such that $\langle Tv, v \rangle = 0$

for all $v \in V$. Show that T = 0. 5

Group B

(Answer **any two** questions) : 6x2=12

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

   $T(x_1, x_2, x_3) = (x_1 + 3x_2 + 2x_3, 3x_1 + 4x_2 + x_3, 2x_1 + x_2 - x_3)$

   Find the dimension of the range space of T^2 and the dimension of the null space of T^2. 6

6. What is minimal polynomial of a square matrix? Find the minimal polynomial of the matrix

   $$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$ 2+4

7. What is a linear functional? Let ϕ be a linear functional on an inner product space V. Show that there is unique vector $v \in V$ such that $\phi(u)=\langle u, v\rangle$

   for every $v \in V$. 1+5

8. Let $T : P_3 [0, 1] \to P_2 [0, 1]$ be defined by $T(p)(x) = p''(x) + p'(x)$. Then find the matrix represented by T with respect to the bases $\{1, x, x^2, x^3\}$ and $\{1, x, x^2\}$ of $P_3 [0, 1]$ and $P_3[0, 1]$ respectively. 6

Group C

(Answer **any four** questions) : 4x2=8

9. a) Is there a linear mapping $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $T(1, 0, 3) = (1, 1)$ and $T(-2, 0, -6) = (2, 1)$? Justify your answer.

b) Let V = c ([0, 1]), the vector space of real valued continuous functions an [0, 1] and define

$$\langle f, g \rangle = \int_0^{1/2} f(t)g(t)dt.$$

Is this an inner product on V?

c) Define Jordon matrix with an example.

d) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$ find Null space & image of linear mapping.

e) Prove that if A and B are similar matrices then they have same characteristic values.

f) Find the symmetric matrix which corresponds to the quadratic form $xy + y^2$.

g) Let V be an inner product space. Suppose $u, v \in V$ are such that $\|u\| = 3, \|u+v\| = 4, \|u-v\| = 6$.

Find $\|v\|$.

h) Is there an inner product on $\mathbb{R}^2$ such that the associated norm is given by

$$\|(x_1, x_2)\| = |x_1| + |x_2| \ ?$$