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End Semester Examination of Semester–I, 2014
Subject : MATHEMATICS (PG)
Paper : 101 (Real Analysis) (Theory)
Full Marks : 40
Time : 2 Hrs

The figures in the margin indicate the marks corresponding to the question

Candidates are requested to give their answers in their own word as far as practicable.

Illustrate the answers whenever necessary

Group A

(Answer **any two** questions) : 10x2=20

1. a) Let $f : [a, b] \rightarrow \mathbb{R}$ be defined by

$$f(x) = \alpha \quad \text{if } x \text{ is rational in } [a, b]$$
$$= \beta \quad \text{if } x \text{ is irrational in } [a, b]$$

where $\alpha \neq \beta$. Show that f is not a function of bounded variation on [a, b]. 5

b) Prove that any finite sum can be expressed as RS-integral. 5

2. a) Prove that every bounded measurable function on [a, b] is Lebesgue integrable on [a, b]. 6

b) Evaluate the Lebesgue integral $\int_{-2}^{5} f(x)\,dx$

where, f(x) $= 3, \quad -2 \leq x \leq 0.$

$= 0, \quad x \leq 0 \leq 1.$

$= 2, \quad x = 1$

$= 4,$ x is rational in (1, 2)

$= -5,$ x is irrational in (1, 2) 4

3. a) Let $U \leq R^n$ be open and connected. Let $f : U \rightarrow R^m$ be differentiable and $Df(x) = 0$ for all $x \in U$. Then show that f is a constant function. Is connectedness of U necessary for the result? Justify. 4+2

b) Let $f : U \leq R^n \rightarrow R^m$ be differentiable at $x \in U$. Then show that the directional derivative $D_v f(x)$ exists for all $v \in R^n$ and $D_v f(x) = Df(x)(v)$. 4

4. a) Let f maps an open set $U \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$, and f be differentiable at a point $x \in U$. Show that the partial derivatives $(D_j f_i)(x)$ exist, and

$$Df(x)(1_j) = \sum_{i=1}^{n} (D_j f_i)(x) x_j u_i \quad (1 \leq j \leq m).$$ 6

b) Show that the set of all invertible linear operators on $\mathbb{R}^n$ is an open subject of $\mathcal{L}(\mathbb{R}^n)$. 4

Group B

(Answer **any two** questions) : 6x2=12

5. Show that the set of invertible n x n matrices over R is open in $m_n(R)$. Is it dense? 4+2

6. Using the Fundamental theorem of Calculus prove that $\iint_R g_x dxdy = \int_{\partial R} gdy$ where R is a square in the plane and f, g : $R^2 \to R$ are smooth. 6

7. a) Evaluate the R.S. integral : 4

$$\int_1^4 (2x+1)d([x]-x)$$

7. b) State Fubine's theorem for two-dimensional Riemann integration. 2

8. a) Let U be an open subject of $\mathbb{R}^n$. Let x ∈ U, and v ∈ $\mathbb{R}^n$. Define directional derivative for the map f : U → $\mathbb{R}^m$ at x in the direction v. 2

b) Show that the function f : $\mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \frac{x^3 y}{x^4 + y^2} \quad \text{if } (x, y) \neq (0, 0)$$

$$= 0 \quad \text{if } (x, y) = (0, 0)$$

has $D_u f(0, 0) = 0$ for all u ∈ $\mathbb{R}^2$ but is not differentiable at (0, 0). 4

Group C

(Answer **any four** questions) : 4x2=8

9. a) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f((x, y)) = (x^2, y^2)$. Is f differentiable? Find the derivative matrix [Df(1, 1)]

b) State Inverse Mapping Theorem.

c) Show by an example that a Lebesgue integrable function may not be Riemnn integrable.

d) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that T is differentiable and find the derivative.

e) Let $f : \mathbb{R}_n - \{0\} \to \mathbb{R}$ be given by

$f(x) = \|x\| = \sqrt{\langle x, y \rangle}$

If f differentiable?

f) Let $f : U \subset R^n \to R$ be a function. Let $a \in U$. What do you mean by that f has a local maximum at 'a'?

g) Evaluate the RS-Integral $\int_1^4 (2x+1)d([x]-x)$

h) Show that an open subject of R is measurable.